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AMTI

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THE ASSOCIATION OF MATHEMATICS TEACHERS OF INDIA

The Association of Mathematics Teachers of India (AMTI) was started in 1965 for the promotion of efforts to improve Mathematics education at all levels. A major aim of the Association is to assist practising teachers of Mathematics in schools in improving their expertise and professional skills. Another important aim is to spot out and foster Mathematical talents in the young. The Association also seeks to disseminate new trends in Mathematics education among parents and public. Other activities of the Association include consultancy services to schools in equipping the Mathematics section of their libraries, in organising children's Mathematics clubs and fairs, in setting up teacher centres in schools, in conducting Mathematics laboratory programmes, in holding practical tests in Mathematics in assisting children in participating investigational projects.

The Association holds " National Mathematical Talent Search Competition " annually and organizes Orientation Courses, Seminars and Workshops for teachers and courses for talented students. A national conference is held annually in different parts of the country for teachers to meet and deliberate on important issues of Mathematics education. Innovative teacher award has been instituted to give public recognition to enterprising and pioneering teachers of Mathematics for which entries from teachers are invited.

An award for contributions to the Mathematics Teacher relating to History of Mathematics in the context of mathematics education had been instituted by Prof.R.C.Gupta.

"The Mathematics Teacher (India)" (MT) is the official quarterly journal of the Association and is issued twice a year. It has been approved for use in schools and colleges of education by the Government departments of education in many States. Besides MT the Association also brings out Junior Mathematician (JM), three issues in a year, especially for school students in English and Tamil.

The membership of the Association is open to all those interested in Mathematics and Mathematics Education. The membership fee inclusive of subscription for "**The Mathematics Teacher (India)**" and effective from April 1993 is as follows:

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Life	Rs. 500	Rs. 1000
Annual (Ordinary)	Rs. 50	—
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The Journal "The Mathematics Teacher" will be supplied free to all members.
Fifty or more subscriptions to Junior Mathematician will entail 20% discount.

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The Mathematics Teacher
(INDIA)

OFFICIAL JOURNAL OF THE ASSOCIATION OF MATHEMATICS TEACHERS OF INDIA

Volume: 47 Issues: 3 & 4 Year: 2011

Prof. Hemalatha Thyagarajan
Editor

EDITORIAL

Welcome to the issue of Mathematics Teacher 47(3&4). This issue has very interesting articles on varied topics. "Knowing the World better Through Mathematics" by Arindam Bose and "Mathematics Stor(e)y" by Sadagopan Rajesh are regarding teaching of Mathematics. The first deals with Mathematical concepts as seen by tribals who have no formal education but who have their own ways of dealing with out of the way calculations. Their thought process is indeed very different and may be used in class room teaching. Rajesh uses story telling as a mode of teaching fractions. He introduces comparison of fractions using three fictional characters in a story setting.

"A Note on Digital Roots" by P.V. Satyanarayana Moorthy and "A Note on the Circum-circles Associated with a Rectangle" by Subramanyam Durbha will be very useful

for teachers and High School students alike. "Brahmagupta's Contributions to Algebra" by Professor A.K.Datta, gives a detailed account of the strides that India had made in the field of Algebra, much before European Mathematicians, with particular reference to Brahmagupta. Professor Bhatia's talk in the 45^{th} Annual Conference, reproduced here, gives the application of Eigen Values to Vibrations. Prof. D.K. Sinha analyses the characteristics of Applied Mathematics and Swaminathan provides some useful tips to introduce Mathematical Concepts in his article. A report on the 45^{th} Annual AMTI Conference is also included.

We request the readers to submit articles on Teaching Methodology in High School and College Mathematics.

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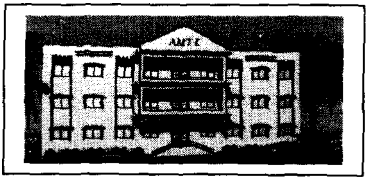

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VIBRATIONS AND EIGENVALUES

Rajendra Bhatia
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Vibrations occur everywhere. My speech reaches you by a series of vibrations starting from my vocal chords and ending at your ear drums. We make music by causing strings, membranes, or air columns to vibrate. Engineers design safe structures by controlling vibrations.

I will describe to you a very simple vibrating system and the mathematics needed to analyse it. The ideas were born in the work of Joseph-Louis Lagrange (1736-1813), and I begin by quoting from the preface of his great book *Méchanique Analitique* published in 1788:

We already have various treatises on mechanics but the plan of this one is entirely new. I have set myself the problem of reducing this science [mechanics], and the art of solving the problems pertaining to it, to general formulae whose simple development gives all the equations necessary for the solutions of each problem ... No diagrams will be found in this work. The methods which I expound in it demand neither constructions nor geometrical or mechanical reasonings, but solely algebraic [analytic] operations subjected to a uniform and regular procedure. Those who like analysis will be pleased to see mechanics become a new branch of it, and will be obliged to me for having extended its domain.

Consider a long thin tight elastic string (like the wire of a *veena*) with fixed end points. If it is plucked slightly and

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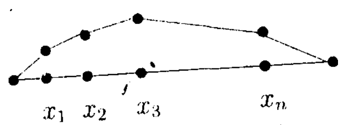
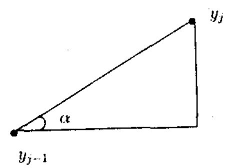
Presidential Address delivered by Prof. Rajendra Bhatia, at the 45 th Annual Conference of AMTI held at Kolkata, during December 27-29, 2010.

released, the string vibrates. The problem is to find equations that describe these vibrations and to find solutions of these equations. The equations were first found by Jean d' Alembert, and two different forms of the solution were given by him and by Leonhard Euler.

Lagrange followed a different path: he *discretised* the problem. Imagine the string is of length $(n+1)d$, has negligible mass, and there are n beads of mass m each placed along the string at regular intervals d :



The string is pulled slightly in the y-direction and the beads are displaced to positions $y_1, y_2, \ldots, y_n$. The tension T in the string is a force that pulls the beads towards the initial position of rest. Let α be the angle that the string between the $(j-1)$th and the jth bead makes with the x-axis:



Then the component of T in the downward direction is $T \sin \alpha$. If α is small, then $\cos \alpha$ is close to 1, and $\sin \alpha$ is close to $\tan \alpha$. Thus the downward component of T is approximately

$$T \tan \alpha = T \, \frac{y_j - y_{j-1}}{d}.$$

Similarly the pull exerted on the jth bead from the other side of the string is

$$T \, \frac{y_j - y_{j+1}}{d}.$$

Thus the total force exerted on the jth bead is

$$\frac{T}{d} \left(2y_j - y_{j-1} - y_{j+1}\right).$$

By Newton's second law of motion

$$\text{Force} = \text{mass} \times \text{acceleration},$$

this force is equal to $m\ddot{y}_j$, where the two dots denote the second derivative with respect to time. So we have

$$m\,\ddot{y}_j = \frac{-T}{d}\left(2y_j - y_{j-1} - y_{j+1}\right). \tag{1}$$

The minus sign outside the brackets indicates that the force is in the "downward" direction. We have n equations, one for each $1 \le j \le n$. It is convenient to write them as a single vector equation

$$\begin{bmatrix} \ddot{y}_1 \\ \ddot{y}_2 \\ \vdots \\ \ddot{y}_n \end{bmatrix} = \frac{-T}{md} \begin{bmatrix} 2 & -1 & & & & \\ -1 & 2 & -1 & & & \\ & -1 & \vdots & & & \\ & & & \vdots & & \\ & & & & \vdots & -1 \\ & & & & -1 & 2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}, \tag{2}$$

or as

$$\ddot{\boldsymbol{y}} = \frac{-T}{md}\,L\,\boldsymbol{y}, \tag{3}$$

where $\boldsymbol{y}$ is the vector with n components $y_1, y_2, \ldots, y_n$, and L is the $n \times n$ matrix with entries $\ell_{ii} = 2$ for all i, $\ell_{ij} = -1$ if $|i - j| = 1$, and $\ell_{ij} = 0$ if $|i - j| > 1$. (A matrix of this special form is called a *tridiagonal* matrix.)

Let us drop the factor $-T/md$ (which we can reinstate later) and study the equation

$$\ddot{\boldsymbol{y}} = L\,\boldsymbol{y}. \tag{4}$$

We want to find solutions of this equation; i.e., we want to find $\boldsymbol{y}(t)$ that satisfy (4). In this we are guided by two

considerations. Our experience tells us that the motion of the string is oscillatory; the simplest oscillatory function we know of is $\sin t$, and its second derivative is equal to itself with a negative sign. Thus it would be reasonable to think of a solution

$$\boldsymbol{y}(t) = (\sin \omega t)\boldsymbol{u}. \tag{5}$$

If we plug this into (4), we get

$$-\omega^2(\sin \omega t)\boldsymbol{u} = (\sin \omega t)L\boldsymbol{u}.$$

So, we must have

$$L\boldsymbol{u} = -\omega^2\boldsymbol{u}.$$

In other words $\boldsymbol{u}$ is an *eigenvector* of L corresponding to *eigenvalue* $-\omega^2$.

So our problem has been reduced to a problem on matrices: find the eigenvalues and eigenvectors of the tridiagonal matrix L. In general, it is not easy to find eigenvalues of a (tridiagonal) matrix. But our L is rather special. The calculation that follows now is very ingenious, and remarkable in its simplicity.

The characteristic equation $L\boldsymbol{u} = \lambda\boldsymbol{u}$ can be written out as

$$-u_{j-1} + 2u_j - u_{j+1} = \lambda u_j, \quad 1 \le j \le n, \tag{6}$$

together with the *boundary conditions*

$$u_0 = u_{n+1} = 0. \tag{7}$$

The two conditions in (7) stem from the fact that the first and the last row of the matrix L are different from the rest of the rows. This is because the two endpoints of the string remain fixed—their displacement in the y-direction is zero. The trigonometric identity

$$\begin{aligned}\sin(j+1)\alpha + \sin(j-1)\alpha &= 2\sin j\alpha \cos\alpha \\ &= 2\sin j\alpha(1 - 2\sin^2\frac{\alpha}{2}),\end{aligned}$$

after a rearrangement, can be written as

$$-\sin(j-1)\alpha + 2\sin j\alpha - \sin(j+1)\alpha = (4\sin^2\frac{\alpha}{2})\sin j\alpha. \quad (8)$$

So the equations (6) are satisfied if we choose

$$\lambda = 4\sin^2\frac{\alpha}{2}, \quad u_j = \sin j\alpha. \quad (9)$$

There are some restrictions on α. The vector $\boldsymbol{u}$ is not zero and hence α cannot be an integral multiple of π. The first condition in (7) is automatically satisfied, and the second dictates that $\sin(n+1)\alpha = 0$. This, in turn, means that $\alpha = k\pi/(n+1)$. Thus the n eigenvalues of L are

$$\lambda = 4\sin^2\frac{k\pi}{2(n+1)}, \quad k = 1, 2, \ldots, n. \quad (10)$$

You can write out for yourself the corresponding eigenvectors.

What does this tell us about our original problem? You are invited to go back to ω and to the equation (3) and think. A bit of "dimension analysis" is helpful here. The quantity T in (3) represents a force. So its units are $\frac{\text{mass}\times\text{length}}{(\text{time})^2}$. The units of $\frac{T}{md}$ are, therefore $(\text{time})^{-2}$. So, after the factor $\frac{-T}{md}$ is reinstated, the quantity ω represents a frequency. This is the frequency of oscillation of the string. It is proportional to $\sqrt{T/md}$. So it increases with the tension and decreases with the mass m of the beads and the distance d between them. Does this correspond to your physical experience?

We can go in several directions from here. Letting d go to zero we approach the usual string with uniformly distributed mass. The matrix L then becomes a differential operator. The equation corresponding to (3) then becomes Euler's equation for the vibrating string. We can study the problem of beads on a *heavy* string. Somewhat surprising may be the fact that the same equations describe the flow of electricity in telephone networks.

The study of the vibrating string led to the discovery of Fourier Series, a subject that eventually became "harmonic analysis", and is behind much of modern technology from CT scans to fast computers.

I end this talk by mentioning a few more things about Lagrange. Many ideas in mechanics go back to him. It has been common to talk of "Lagrangian Mechanics" and "Hamiltonian Mechanics" as the two viewpoints of this subject. Along with L. Euler he was the founder of the *calculus of variations.* The problem that led Lagrange to this subject was his study of the *tautochrone,* the curve moving on which a weighted particle arrives at a fixed point in the same time independent of its initial position. The Lagrange method of undetermined multipliers is one of the most used tools for finding maxima and minima of functions of several variables. Every student of group theory learns Lagrange's theorem that the order of a subgroup H of a finite group G divides the order of G. In number theory he proved several theorems, one of which called "Wilson's theorem" says that n is a prime if and only if $(n-1)!+1$ is divisible by n. In addition to all this work Lagrange was a member of the committee appointed by the French Academy of Sciences to standardise weights and measures. The metric system with a decimal base was introduced by this committee.

WHITHER EVOLVING CHARACTERISTICS OF APPLIED MATHEMATICS ?

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1. Prefatory remarks

The write-up of this presentation makes me reminiscent of what I wrote in the Bulletin of AMTI, way back in 1979 as a tribute to Professor P.L.Bhatnagar (PLB). The trio- AMTI, Prof PLB, and myself - had the convergence in 1972 at the 2nd Annual Conference of AMTI and 7th Annual Conference of AMTI in Calcutta. There is hardly any need of repetition on PLB. It would be fair enough now to pay homage to his memory to the extent it fits in with the current realities. PLB used to be described as an applied mathematician. Earlier days could talk about few pockets of pursuits in applied mathematics, Calcutta, Roorkee, Allahabad, Benaras, Waltair and Bangalore, with the last one being steered through the leadership of PLB. His early mathematical mainstay being in Allahabad, and with a short stint of research work under the supervision of Professor B N Prasad, he moved out for research in applied mathematics with Prof. A C Banerji, an alumnus of Calcutta University. PLB became later a doyen in Applied Mathematics; he had his birthplace somewhere in Rajasthan, which I could know when he was Vice Chancellor of Rajasthan University, Jaipur. His interests in mathematics education, particularly at the school level, have been known, not only since 1972 but at

Prof. P.L.Bhatnagar Memorial Lecture delivered at the 45th Annual Conference of AMTI held at Kolkata, during December 27-29, 2010.

Jaipur as well, at the NCERT Study Group meeting, which was chaired by Professor D S Kothari, who also hailed from Kota of Rajasthan. PLB's work on 'ionized gases' in the realm of Plasma Physics is cited as his original contribution. I had the benefit of interaction with him at Himachal Pradesh University, where he spent sometime as its Vice-Chancellor.

He became a member of UPSC and I had an opportunity to work with him there as well, particularly in renewal of mathematics curricula for competitive examinations of UPSC. His leadership for creation of the Mathematical Institute at Bharwari, close to Allahabad, has to be reckoned with, as a precursor of Harish Chandra Institute now based at Allahabad. The UNESCO and ICMI (International Commission of Mathematical Instruction) sponsored Workshop at Bharwari Institute; this was inaugurated by Professor S Nurul Hasan, Union Education Minister. The Workshop and the publication of its Proceedings, still come up vividly in the corridors of my memory. His participation in the ICME (International Congress on Mathematics Education) at Karlsruhe, FR Germany is still green in my memory. His book dealing with Matrices and Transformations ought to be referred not just as textual material but as something going well beyond a wide variety of takers. That mathematics education occupied a distinctive space in his realm of educational activities is amply testified by such events.

I feel privileged to deliver this lecture; indeed, I have delivered the P.L.Bhatnagar Memorial Lecture in one of the Annual Conferences of Indian Mathematical Society which had its origin in Madras (now Chennai). Yours is a community of mathematical practitioners that keeps on arranging annually and providing the lecture dedicated to a crusader during his period of Applied Mathematics.

2. Synopsis

Any lecture dedicated to the memory of Late Professor P.L.Bhatnagar has to cite what he sought to achieve through his works in applied mathematics. One should also recall his thoughts about possible implications of applied mathematics in the realm of mathematics education. To my perception, what seems to be appropriate for presentation to a gathering like this is how one can situate him in the wider context of trends in the pursuits of applied mathematics. Of late, one finds a variety of vocabularies that come very close in the neighbourhood of the much-talked about "applied mathematics". That the adjectival use of the word "applied" to any area of mathematics gives rise to credibility and usability has to be stressed. Applicable mathematics has already come into vogue. Information and Communication (Technology) have made a considerable dent in mathematics - pure, applied or with applications. A part of this lecture will touch upon some of these aspects. Applied mathematics can in no way be distanced off from the usages necessitated by technology and management. Applied Mathematics, whatever might be its origin, has built up many evolving trajectories because of complexity of realities involved. This presentation seeks to take on some pressing issues of mathematical realities in the wider perspectives of problem-solving. Mathematical sciences or the like keep on emerging because of inroads of mathematics in various disciplines so as to be new providers of better insights and understanding and this will also be briefly focused here.

3. Introductory remarks

As already indicated, this lecture ought to hover around developments in applied mathematics, particularly the emerging areas. Any conspectus on such lines cannot dispense with the history of applied mathematics, one can

trace back its origin to the days of Newton, Laplace, Lagrange, D'Alembert, Euler, Poincare and many celebrities in the arena of mathematics. Mechanics happened to be the forte of most of them, deriving the motivation largely from celestial aspects. European ouvre appeared to have dominated immensely this kind of mathematics. Mathematical enterprises, in those days, (and also as of now) did not put up basics between 'pure' and 'applied' mathematics. That geometrical endeavours stemmed from run-of-the-mill requirements, do not need to be considered. 'Mixed mathematics' could be the middle-of-the-path designation that was resorted to so that pursuits in 'pure physics' could be accommodated under such an umbrella. The Department of Applied Mathematics and Theoretical Physics in Cambridge continues to be a strong pointer in this direction. The word 'applied' pertaining to a discipline has acquired several connotations which need to be delved into so that implications of the same for mathematics, in particular, can be fathomed. Users of mathematics *per se* are often skeptic about such usages. Here the purpose is to focus on areas of mathematics that are contextually significant and conceptually rich so as to be labeled as 'applied mathematics' worth the name. As mentioned in the last few lines of the Synopsis, variants of 'mathematical sciences' will be dealt with, so that new pathways of 'applied mathematics' can be identified, along with evolving forms.

4. Classical areas of Applied Mathematics: A brief resume

Whatever be the genesis, few areas of 'applied mathematics' have come to stay. Mechanics of particles and rigid systems have already been mentioned as areas of engagement of mathematics-celebs. The problems there, can in no way be regarded as ritualistic. The solutions of problems there, may have given rise not merely to twists and turns of

mathematical acrobatics but to new mathematical ideas, too; otherwise, elliptic functions, optimization (brachistochrone) etc. would not have come into existence. The complexions of physics e.g. geometrical physics warranted the use of mathematical techniques. There is no dearth of exemplars in this direction. In a way, geometrization or in wider terms, mathematization took off long ago. Solid mechanics and fluid mechanics are recognizably ongoing areas of pursuits in different levels; Lunar Dynamics and Marine Dynamics did feature as subjects of study in some well-equipped establishments. Elasticity, plasticity, viscoelasticity, thermoelasticity and the like could rope in many researchers over the decades; so is the flourishing of fluid dynamics, visco-fluid dynamics, aerodynamics, ballistics etc. Theoretical Seismology, Geodesy and Geophysics and Meteorology must have contributed enormously. European Consortium of Mathematics in Industry (ECMI) with its origin in FR Germany, can be taken as a spinoff for mathematizing not just applying extant technologies in mathematics. Equally important are ventures that sprang into existence because of proliferating methods of numerical analysis followed by mathematically oriented computational studies; of course, prior to the beginning of in-depth software and programming studies.

Studies in Electricity and Magnetism, belong to the domain of applied mathematics as testified by classical works of Maxwell. While Britishers followed it up assiduously, for example, by James Jeans, the corresponding European realms used to be led by protagonists of field theory, for example, Einstein, Sommerfield, Courant, Max Born, essentially on classical German styles. Plasma Physics, Space Dynamics and Astrophysics had their roots in basics of applied mathematical entities. Interdisciplinary trends and theorization thereof, with the aid of mathematical tools,

could become discernible, paving the way for exercises in these directions. Magnetofluiddynamics, electromechanical and magnetomechanical studies (such as piezoelectricity, magnetostriction, electrostriction, elastic dielectrics etc,) are fast assuming characters which purely classical peers on applied mathematics could not envisage.

5. Rationale for further developments

The foregoing lines show, in substantial terms, that the need for mathematization could be increasingly realized. Problems belonging to seemingly non-mathematical areas are used to be drawn upon. The familiar disciplines like Physical Sciences could be amenable to mathematization so that some sort of prediction could be made in regard to the situation concerned. Phenomena have kept on happening and so do many fields of human activity. For instance, D'Anconna's observation on species-interaction in the seas rushed him to his father-in-law Vito Volterra, a mathematician, who could establish mathematically the realities observed by his son-in-law. The well-known Lotka-Volterra system from the mathematical. standpoint, could thus be born. It forms, undeniably, the basis of mathematical ecology. Mathematical Biology could have come up with a greater significance because of applications of reduction hypothesis. It was due to Rashevsky, a Russian biophysicist, that Mathematical Biology could move forward in mathematically rigorous ways. The hierarchies of transformations had been accommodated. Mathematical Biosciences has acquired a genre; so has Mathematical Sociology, Mathematical Social sciences. One talks about Mathematical Learning Theory. Mathematical Ecology has kept on flying its wings vigorously because of necessities in environmental systems. Biological, Social, Ecological , Ethnic systems could hardly proceed on their own to be able to predict the futurities of phenomena in those arenas. Application of

mathematical techniques has been a great provider for coping with problems in such areas. Modelling of the system has become a necessity.

6. Glimpsing through 'means' in Applied Mathematics

In any problem-solving exercise, 'solution' of a problem appears to be the 'end'. 'Means' to achieve the 'end', expectedly ought to be many. But shouldn't, in mathematical parlance, one be concerned with existence of anything worthwhile to deal with? 'Existence' of a problem in realms of applied mathematics, is fairly well taken by 'posing' the situation and to begin with, in nonmathematical language. Of course, with a bit of fineness, 'ill-posed' and 'well-posed' are two terms to be looked for. Next comes necessarily a process, entitled 'formulation', couched step by step, in mathematical language consisting of symbols and their representations. Then ensue uses of mathematical operations leading to some results which await interpretation. For example, in the study of ecosystems, the behaviour of single species population is described by

$$\frac{dN}{dt} = (b - d)N \tag{1}$$

where N is the population size, b, d being constant birth and death rates. That's a situation which assumes a lot. This leads to

$$N = N_0 e^{rt} \tag{2}$$

where r (= b-d) is the intrinsic potential. (1) is modified to assume the form

$$\frac{dN}{dt} = (a - bN)N \tag{3}$$

where a and b are constants, assuming r to be a function of N. in accordance with the reality. Here is a simple differential equation and a simple solution, which, when graphed, shows a S-curve; this indicates a saturation limit (a/b) of population

size. It has environmental realities. Here are examples of one kind of means as a provider of insights into the population growth (or decay) phenomenon. The Lotka-Volterra system, if they are to be expressed in terms of differential equations, is given by

$$\frac{dN_1}{dt} = (a_1 - b_1 N_2)N_1 \tag{4}$$

$$\frac{dN_2}{dt} = (-a_2 - b_2 N_1)N_2)$$

where a 's , b 's are constants, N_1 , N_2 are sizes representing prey/host and predate/parasite populations. These are far from being linear ; indeed, they are similar. The 'means' here ought to be different. Prior to a solution per se. whatever be the techniques, one can obtain a qualitative view of the situation. These, thus, appear as situations that require not merely quantitative results for interpretation but also qualitative analysis for a wider context. What are the pathways of possible solutions? How can they be characterized? Nonlinear science would not have come up but for the contributions of the Russian Mathematician Alexander Mikhailovish Liapunov who brought in the idea of 'stability' of systems in inimitable ways. The simple dynamics on the non-linearity on the motion of a pendulum brings up some revelations which, if portrayed, shows a variety of characteristic features, having the vocabularies like saddle point, mode, center, focus etc. Qualitative behaviour of linear systems around critical points is worth studying. Besides differential equations, the above Pearl-Veraulst logistic equation (3) has the form

$$N_{t+1} = \{(a+1) - bN_t\}. \tag{5}$$

Orbits or trajectories keep on evolving and thus, the vocabularies 'Limit Cycles', 'Catastrophes' and 'Bifurcations' form the core part of qualitative studies born from differential topology, having come up in mathematical literature because

of Rene Thom and Christopher Zeeman. Manifolds do feature abundantly in such studies. Thompson *etal* make use of structured stability of engineering systems with such themes. Dynamical systems have come to stay; evolving features; limit-cycles of systems cannot but be sought for in nonlinear systems, wherever they may occur, in physical, biological or social sciences. Buckling in mechanics reappears through qualitative motions, namely, Hopf-bifurcation(s).

The consideration of Information and Communication Technology (ICM) dimensionalizes mathematics in totally novel ways. New variants of algebra and geometry appear in the technological world; for example, the concept of mapping. Shouldn't we explore whether a software environment has a dynamics or not? Does a computerized environment provide a new design of mathematical activity? What can digital technologies provide for pursuits in mathematics? Would these allow, new formats of applied mathematics to emerge?

7. Concluding remarks

No discussion on applied mathematics can be complete without referring to mathematical modeling. Not that the structure of modeling is to be indicated but it has not been essentialized with dynamical attributes. Applied mathematics does intervene to formulate a model of the situation in realistic terms. Data are to be collected appropriately, so that applied mathematics acquires its ownership. That may be at stake, if data, relations, object and conditions are not attended to. Mathematics has to have, thus, several phases. Credit-worthiness is a continuing concern of any applied mathematics phenomenon. There exists in such phenomena a dynamicity of environment which necessitates a back and forth movement, once one steps out of the mathematical reality. Undeniably, a sort of interpretation then

sets in. Situation and situativity occupy leading heights in such agile motions. Mathematical Biology/Ecology, Mathematical Sociology, Mathematical Psychology etc. are mathematics of necessity of the discipline concerned while Biomathematics, Sociomathematics etc. necessitate the creation of new concepts and techniques in the mathematical domain. But applied functional analysis, applied algebra, applied topology etc. connote newly acquired dimensions, based on innovative approach of the disciplines concerned. Cryptography, as is well-known, makes most vast use of algebra and number theory. Practitioners of applied mathematics are kept on their toes if the hiatuses between three realities, pre-, in situ and post-mathematics are missed in one way or the other. Bridges between inside and outside of situation need to be established, through classrooms, machines and often, through thinking aloud or through dialogues. Fuzziness, in such contexts, has to be reckoned with all the formalizations around trajectories of applied mathematical exercises. Intervals of confidence cannot be built up, unless concepts and techniques, in realms of applied mathematics, are allowed to be through Socratized dialogues. Characteristics ought to emerge, as one phase paves the way for the other. Here, one may have a naive beginning and naivete keeps on evolving so as to rein in a qualitative viewpoint, despite conflict or a cooperative approach, aiming for a solution; then its aftermath impinges on the original statement of mathematics. Glancing through textual stuff in applied mathematics can hardly cease to be a worthwhile task.

BRAHMAGUPTA'S CONTRIBUTIONS TO ALGEBRA

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Introduction

In a monograph on "The Witt Group and The Problem of Luroth" published in 1990, Manuel Ojanguren begins a chapter (Chapter 5) by quoting two Sanskrit verses (in Devanāgari script) from Chapter 18 of Brāhma Sphuṭa Siddhānta, a treatise composed in 628 CE by Brahmagupta. The chapter itself is titled "**Also sprach Brahmagupta**" — "Thus Spake Brahmagupta"!

What is so special about the two verses of Brahmagupta? They describe a principle which is the starting point of an area in modern algebra. Reformulated in modern language, the verses yield a basic result in the area of "Quadratic Forms" which was generalised by Pfister in 1965 (not a typing error — it is 1965). Pfister's discovery opened up a new frontier in algebra. The first result (5.1) of Chapter 5 of Ojanguren's book is (a reformulation of) Brahmagupta's result; the second (5.2) being Pfister's.

We can get glimpses of the greatness of Brahmagupta's work by noting such responses by modern mathematicians in general and algebraists in particular. In a lecture in Bangalore in October 1993, on as broad a topic as "Mathematics as a basic science", Michael Atiyah, one of the greatest mathematicians

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of our time, remarked (Current Science, Dec. 1993, p 913):

> "Number Theory for its own sake, as a great intellectual challenge, has a long history, particularly here in India. Already in the 7th century, Brahmagupta made important contributions to what is now known (incorrectly) as Pell's Equation."

However most of us, including professional mathematicians, seem to be oblivious and indifferent to these facts of history. We act as if Āryabhaṭa. Brahmagupta, Bhāskarācārya, Mādhavācārya, etc, are mythological names, not historical figures with concrete contributions to mathematics and astronomy.

In this article, we shall mention some of Brahmagupta's contributions to algebra. First, let us place Brahmagupta in the standard chronological framework of Indian history.

Brahmagupta's Time, Place of Work, Extant Texts

Brahmagupta was born in 598 CE, during the Classical Age of post-Vedic ancient Indian history. The Classical Age was a period of robust intellectual activity, a period of efflorescence in philosophy, literature, science, polity, art and architecture. Among the predecessors of Brahmagupta, one may recall three geniuses: Samudragupta (c.350 CE), the statesman, Kālidāsa, the poet-playwright and Āryabhaṭa (c.500 CE), the astronomer-mathematician. Brahmagupta composed his magnum opus Brāhma Sphuṭa Siddhānta in 628 CE during the reign of King Vyāghramukha of the Cāpa dynasty who was known as an "ornament of the Cāpa race". He was thus a contemporary of Emperor Harshavardhana (who reigned during 606-647 CE).

Brahmagupta's place of work was Bhīnmāla (earlier Bhillamāla). located in the Jalore district of Rajasthan (close

to north Gujarat). He was famous as "Bhillamālakācārya". Bhīnmāla was once a premier city of northern India and a great centre of learning. It is said to be the birthplace of the 7th century Sanskrit poet Māgha. the author of Śiśupālavadha.

Brāhma Sphuṭa Siddhānta (628 CE) is a treatise in astronomy and mathematics comprising 1008 verses. Prior to this text, Brahmagupta authored a mathematics text Dhyānagraha (72 verses). Brahmagupta later wrote another work Khaṇḍa Khādyaka in astronomy (194 verses) around 665 CE.

In astronomy, Brahmagupta laid emphasis on direct observations with instruments. He made several improvements over his predecessors. He gave a more accurate method of calculating the true longitude of the sun and the moon. This is especially important at the time of a solar eclipse.

Mathematics in India attained one of its highest peaks due to the brilliant and versatile Brahmagupta. Of the 24 chapters (1008 verses) in Brāhma Sphuṭa Siddhānta, the 12th chapter (66 verses) covers topics like arithmetic, geometry and mensuration, while the 18th chapter (102 verses) is on algebra. These two chapters in mathematics were translated into English by H.T. Colebrooke in 1817. Some of Brahmagupta's discoveries in mathematics, recorded in these two chapters, were introduced, understood or rediscovered in Europe only during the 16th–18th centuries, that is, 1000–1200 years after his era.

Brahmagupta refers to the science of algebra as *kuṭṭaka* and his chapter on algebra as Kuṭṭakādhyāyaḥ. Through this chapter, Brahmagupta laid a firm foundation for classical algebra — use of symbols and formulae, formation and solutions of equations, operations with negative numbers and zero; and inculcated a consciousness of algebra. Amazingly,

Brahmagupta's chapter also contains a sophisticated aspect of modern abstract algebra: the formulation and study of composition laws. Brahmagupta discovered a composition law (which is quoted in Ojanguren's book) and applied it to the study of the number-theoretic problem of finding integer solutions to certain equations. This is a truly astonishing achievement from one who had to develop the basics of algebra. We propose to make a brief mention of his masterly touches in diverse aspects of algebra.

Foundations of Algebra

In the Preface of [DS], Datta-Singh remarks:

> "The use of symbols — letters of the alphabet to denote unknowns — and equations are the foundations of the science of algebra."

Let us first see Brahmagupta's role in creating the discipline of algebra from this point of view.

Use of Symbols

One of the landmarks in the history of mathematics, which ensured a rapid acceleration in the progress of the subject, is the introduction of symbols in mathematics presentation. Brahmagupta played an important role in this development. He used the suggestive term *avyakta* to denote one unknown quantity. For an equation involving several unknowns, Brahmagupta prescribes use of "*varṇa*" (which means letter of the alphabet as well as colour) as symbols for different unknowns. Thus the first letter of different colour-names became the symbols for unknowns.

In Sthānāṅga-sūtra (c.300 BCE), the term *yāvat tāvat* is used for an unknown quantity (as well as linear equation in one unknown); this term is used in several later texts. In the expositions of Pṛthūdakasvāmin (c.860 CE), Śrīpati

(1039 CE), Bhāskara II (1150 CE), Nārāyaṇa (c.1350 CE) and others, colour-names like *kālaka* (black), *nīlaka* (blue), *pītaka* (yellow), *lohitaka* (red), *harītaka* (green), etc, were used for other unknowns. Thus *yā* (from *yāvat tāvat*) became the symbol for the first unknown x, *kā* (fr. *kālaka*) the symbol for the 2nd unknown y, *nī* (fr. *nīlaka*, blue) the symbol for the 3rd unknown z, and so on.

Formation of Equations

Brahmagupta expounded principles for forming equations out of the conditions in a given problem. He used the term *samīkaraṇa* or *sama-karana* (making equal), or more simply *sama* for equation, *varṇa* (letter/colour) for different unknowns, *guṇaka/guṇakāra* (multiplier) for coefficients, *rūpa* (appearance) for known (visible) portions of the equation and *gata* for power of a term. There were already terms for the second, third and fourth powers of a number, namely, *varga* (square), *ghana* (cube) and *varga-varga* respectively. Brahmagupta expressed powers higher than the fourth in a scientific manner by adding the suffix *gata* to the name of the number indicating that power: thus, he calls the fifth power *pañca-gata* (i.e., raised to the fifth), the sixth power *saḍ-gata* (raised to the sixth); and so on.

Brahmagupta gave a clear plan for *writing* of equations and then the steps for the clearance of the equations. There are also statements on classification of equations. There is a description of the algebra of polynomials in several unknowns—the principles of formation of polynomials and the principles for their addition and multiplication.

Enlargement of number system

A remarkable feature of Brahmagupta's chapter on algebra is the subsection on positives, negatives, zero and operations

thereof. Brahmagupta brought out the difference between positive and negative numbers by attaching to them the ideas of "possession" and "debt" respectively. Regarding the subtlety of the concept of negative numbers, Felix Klein, a great mathematician of late 19th century, remarked:

> Let us realize once and emphatically how extraordinarily difficult in principle is the step, which is taken in school, when negative numbers are introduced... Here, for the first time, we meet the transition from concrete to formal mathematics. The complete mastery of this transition requires a high order of ability in abstraction.

A momentous step in the development of mathematics was taken by the Indians when they enlarged the number system by introducing negative numbers and gave negative numbers and zero the status of integers on which arithmetic operations like addition, subtraction and multiplication are to be performed. The precise genesis of the two concepts is obscure but it is from Brahmagupta onwards that the principles and rules of operations involving negative numbers and zero are spelt out clearly to become part of basic mathematics.

The mathematical zero is a multi-faceted concept. Two of its most prominent aspects are as: (i) a symbol, with the status of a digit, acting as a place-holder in a place-value numeral system; (ii) an integer in a number-system where one can perform fundamental binary operations like addition and multiplication. Roughly speaking, they are the roles played by zero in arithmetic and algebra respectively. Brahmagupta took the profound step of introducing zero *as an integer* in algebra. He defined zero as $a - a$ and this definition is repeated in all later Indian texts.

Brahmagupta clearly stated all the ring-theoretic rules of operations involving zero and negative numbers. This

includes rules like $0 \times a = 0$, $a + (-b) = a - b$ and the subtler rules involving multiplication of negatives like $(-a)b = -(ab), (-a)(-b) = ab$, etc. To paraphrase in modern language, Brahmagupta had brought out the ring-structure of integers and the role of zero as the additive identity in the ring.

A Historical Perspective

For a proper appreciation of Brahmagupta contributions to the foundations, one could contrast his work (and the work of subsequent ancient Indians) with those of pre-Renaissance Arab and European mathematicians. Prior to the 16th century, the clumsy "rule of false position" was in vogue, among Arab and European mathematicians, for solving any simple determinate linear equation in one variable. The rule may be stated in our current algebraic language as follows: to solve a problem involving a simple linear equation of the type $ax+b=c$ in one unknown x (where a, b, c are *positive* integers), one assumes an arbitrary value, say u, for the unknown x; computes $d = au + b$, and then applies the rule $x = u + (c - d)/a$ or $x = u - (d-c)/a$, depending on whether $c > d$ or $d > c$. The rule may appear weird now, but it was used in a mathematical atmosphere where one did not have a notation for the unknown, did not handle negative numbers, and had not developed the habit of framing and solving algebraic equations.

Negative numbers began to be accepted in Europe only during the 16th century. Although Newton had strongly advocated negative numbers in the 17th century, and gave the rules described by Brahmagupta a millennium back, even as late as 18th century, many English mathematicians had difficulty in accepting negative numbers. For a long time, European mathematicians continued with the Arab tradition of considering *separately* the three types of quadratic equations with positive coefficients, i.e., $ax^2 + bx = c$, $ax^2 + c = bx$, $bx +$

$c = ax^2$. But thanks to the early proficiency with negative numbers, Brahmagupta and subsequent Indian algebraists used to give a unified treatment of quadratic equations. For more details on this history, see [M].

Solutions of Linear and Quadratic Equations

Brahmagupta gave the modern method for solving simultaneous linear equations in several unknowns by, what is now known as, the Gaussian "Method of Elimination".

Brahmagupta *explicitly* described the general formula for the solution of the quadratic equation $ax^2 + bx = c$ where a, b, c are *arbitrary*. Before him, Āryabhaṭa (499 CE) had also described the formula, but the coefficients were quantities in specific problems (for instance, quantities expressed in terms of the time, principal, amount, etc, in a problem on interest). Brahmagupta made effective applications of his formula in astronomy and arithmetic.

Brahmagupta was aware that a quadratic equation has two roots though, in practice, he retained only one root from consideration of utility in his problems. He sometimes used the positive and sometimes the negative sign attached to the radical. He also showed how to solve simultaneous quadratic equations.

We shall discuss Brahmagupta's treatment of indeterminate equations in the last two sections.

Algebraic Skill in Arithmetic, Geometry and Trigonometry

Some of Brahmagupta's contributions in other areas of mathematics can be seen to be a consequence of his strength in algebra. For instance, in his chapter on arithmetic,

Brahmagupta used his algebraic skill to give the useful rule

$$\frac{a}{b} = \frac{a}{b+h} + \frac{a}{b+h}\,\frac{h}{b}$$

which is often convenient in expressing a fraction $\frac{a}{b}$ with $a > b$ as a mixed fraction. For instance, applying the rule, one can quickly see that $\frac{1920}{93} = 20\frac{20}{31}$ or that $\frac{9999}{97} = 103\frac{8}{97}$. To appreciate this rule, one has to keep in mind that fractions had been a source of difficulty in ancient civilizations including the Egyptian and the Greek and that, in Europe, most of the standard rules on fractions became commonplace only in the 17th century.

In geometry, Brahmagupta had given formulae for the area and diagonals of a cyclic quadrilateral which were to be rediscovered in Europe in the 17th century. But apart from stating these theorems of geometry, Brahmagupta had blended his number-theoretic and geometric skill to describe several interesting geometric constructions of triangles and cyclic quadrilaterals whose sides and areas are *rational numbers.*

Brahmagupta made effective use of his solutions for the equation $x^2 + y^2 = z^2$ — the general integral solution $(2mn, m^2 - n^2, m^2 + n^2)$, where m, n are unequal integers, and the general rational solution $(r, \frac{1}{2}(\frac{r^2}{s} - s), \frac{1}{2}(\frac{r^2}{s} + s))$, where r, s are any two rational numbers. Thus, given a rational number r, Brahmagupta described a rational right-angled triangle one of whose sides (other than the hypotenuse) is of length r — the other sides would be of the form $\frac{1}{2}(\frac{r^2}{s} - s), \frac{1}{2}(\frac{r^2}{s} + s)$. Later Mahāvira (850 CE) used Brahmagupta's formula to construct a right-angled triangle with rational sides when the hypotenuse is given. These results were given in Europe much later by Fibonacci (1202) and Viete (1580). Again, juxtaposing two right-angled triangles, Brahmagupta derived, from any two unequal integers m, n with $m > n$, an isoceles traingle with

integral sides $m^2 + n^2$, integral altitude $2mn$ and integral base $2(m^2 - n^2)$. Similarly, from any rational number r, he gave rational isoceles triangles with altitude r, the sides being of the form $\frac{1}{2}(\frac{r^2}{s} + s)$ and base $\frac{r^2}{s} - s$. Brahmagupta also constructed a scalene triangle with rational sides and a given rational altitude t. The sides of this triangle are of the form $\frac{1}{2}(\frac{t^2}{r}+r)$, $\frac{1}{2}(\frac{t^2}{s}+s)$, $\frac{1}{2}(\frac{t^2}{r}+\frac{t^2}{s}-r-s)$. This was again obtained as an immediate corollary of his construction of right-angled triangles — by taking two triangles of the form $(t, \frac{1}{2}(\frac{t^2}{r} - r), \frac{1}{2}(\frac{t^2}{r} + r))$ and $(t, \frac{1}{2}(\frac{t^2}{s} - s), \frac{1}{2}(\frac{t^2}{s} + s))$ and putting these triangles side by side with the side t common and the two right-angles adjacent. This problem and its solution were given in Europe 1000 years later by Bachet (1621).

Brahmagupta gave an ingenious construction of a cyclic quadrilateral in which the sides, diagonals, circum-diameter and area are all integers and the diagonals are perpendicular to each another. Such a quadrilateral is called a Brahmagupta quadrilateral. One takes two triples (a, b, c) and (u, v, w) such that $a^2+b^2 = c^2$ and $u^2+v^2 = w^2$ and forms four right-angled triangles with sides (au, av, aw), (au, bu, cu), (bu, bv, bw) and (av, bv, cv). These triangles are juxtaposed using the pairs of equal sides au, bu, bv, av — then the four right-angles have a common vertex. The quadrilateral formed is cyclic having consecutive sides aw, cu, bw, cv, diagonals $(au+bv)$, $(av+bu)$ and circum-diameter cw. Brahmagupta also described how to construct an isoceles trapezium whose sides, diagonals, altitudes, segments of base and area are all integers, and a similar construction for a trapezium with three equal sides. For more details about such rational constructions, see [DS].

Another result of algebraic flavour occurs in Brahmagupta's treatise Khaṇḍa Khādyaka where he gave an interpolation formula for calculating the sines of intermediate angles from a sine table. His rule is equivalent to the Newton-

Stirling formula up to second-order difference

$$f(a+xh) = f(a) + x\frac{\Delta f(a) + \Delta f(a-h)}{2} + x^2\frac{\Delta^2 f(a-h)}{2}.$$

Brahmagupta is the first to have used second-order difference, again a thousand years before its re-emergence in Europe. Although the interpolation formula is stated in the context of the sine function, the commentators made it clear that his rule applies to any function.

Algebra as a Distinct Important Discipline

While going through ancient Indian mathematics, one gets the feeling that if there was a moment in Indian history when the creative Word from the god of mathematics took the following form and result: "Let there be Algebra and there was Algebra". then it must have been the composition of Brāhma Sphuṭa Siddhānta by Brahmagupta (628 CE). This treatise gives a distinct identity to the subject.

Algebra occurs in an implicit form in the mathematics of several ancient civilisations. In India, the constructive geometry of the *Śulba Sūtras*, the oldest extant mathematics texts, involves algebraic formulae in geometric form. The discoveries of certain algorithms in computational arithmetic are also indicative of a subtle algebraic acumen. Principles and results, which may be identified as belonging to algebra proper, can be found in texts preceding Brāhma Sphuṭa Siddhānta. Brahmagupta's great predecessor Āryabhaṭa (c.500) has several algebraic results and ideas in his pioneering treatise Āryabhaṭīya. There is a term for an unknown quantity, there are solutions of linear and quadratic equations, there are formulae for arithmetic progression and related sums of finite series and there is a subtle method for solving the linear indeterminate equation $ax - by = c$ in integers. Thus many ingredients of basic algebra are already there, mostly placed in

the later half of the mathematics chapter of Āryabhaṭīya. And one may justifiably call Āryabhaṭa as a founder of "algebra" in India for his algebra-results.

But still one is yet to see a conscious formulation of the science of algebra as a *Śāstra*. That leap is taken by Brahmagupta when he defines "*Kuṭṭaka*" (his term for algebra)[1], emphasises its importance and devotes an entire chapter *Kuṭṭakādhyāyaḥ* (chapter on algebra) to the subject.

This chapter begins with a statement regarding the power and the indispensability of the subject and lists its core topics:

> "Since questions can scarcely be solved without algebra, therefore, I shall expound on algebra with examples.
>
> "By knowing well the pulveriser, the zero, negative and positive quantities, unknowns, elimination of the middle term, symbols, equations with one unknown, equations involving products of distinct unknowns and the square-nature [indeterminate quadratic equation $Dx^2 + m = y^2$], one becomes an *acarya* among the learned."

Towards the end of the chapter, he talks about the glory of algebra:

> "As the sun eclipses the stars by his brilliancy, so the man of knowledge will eclipse the fame of others in assemblies of the

[1]Brahmagupta uses the same word *Kuṭṭaka* in two senses: (i) the subject we now call "algebra" and (ii) the (pulverizer) technique for solving a specific problem, the linear indeterminate equation $ax - by = c$, considered a very important topic by Indian astronomer-mathematicians. His terminology may appear to have an element of ambiguity; it is similar to the present situation in higher mathematics: we use "commutative algebra" to sometimes mean a certain subject founded by Emmy Noether and others, and sometimes to mean a specific object of study in that subject — a commutative ring A together with a ring homomorphism $f : R \to A$ from a base ring R to A.

> people if he proposes algebraic problems, and still more if he solves them."

One may justifiably say that Babylonians were doing algebra, the Vedic Indians, Chinese and Greeks were doing geometric algebra; and so on. But they were perhaps not conscious that they were doing algebra. These early developments are remarkable achievements, but a conscious development of a subject is a contribution of a different quality, much more profound in its impact. It is doubtful whether even Diophantus or Āryabhaṭa, both gifted algebraists, had a clear well-structured vision of algebra.

While Āryabhaṭa is the first known savant in ancient India to compose a separate chapter on mathematics in a major astronomy treatise, Brahmagupta (628) is the first (known) to compose a separate chapter on algebra in a treatise. He was a pioneering visionary who recognised the importance of algebra at an early date. Besides, while the foundations of algebra were already laid by the time of Āryabhaṭa, the real technical advances have been due to the genius of Brahmagupta. For several centuries thereafter, Indian mathematicians consciously focussed their energies to the development of algebra.

Brahmagupta's term for algebra *kuṭṭaka* (pulverisation), reminiscent of Fermat's "descent", indicates that it is a process of solving mathematical problems by a process of simplification — by breaking quantities (for instance, the coefficients or solutions of a given equation) into progressively smaller pieces. It captures a sophisticated aspect of the subject.

The cultivation of mathematics in India for a long time seemed to have been motivated by practical necessities including the demands of rituals and philosophy or astronomy. But in the works of Brahmagupta, especially in the chapter on algebra, we find the mind and attitude of a pure mathematician.

While some of his work undoubtedly arose from the needs of astronomy, and Brahmagupta was indeed one of India's greatest astronomers, there is a substantial portion which is very clearly done as pure mathematics — for intrinsic mathematical importance and the sheer joy. As he mentions towards the end of the chapter on algebra:

> "These problems are proposed simply for pleasure; the wise man can invent a thousand others, or he can solve the problems of others by the rules given here."

Solutions of Linear Indeterminate Equations

An algebraic (i.e., polynomial) equation in more than one variable (or more generally, a system of m algebraic equations in more than m variables) is called *indeterminate.* The term is suggestive of the fact that such a system may have infinitely many solutions. Indians investigated the hard and subtle number-theoretic problem of finding all *integer* solutions of various indeterminate equations. As Cajori writes [Ca, p 94–95]:

> "Indeterminate analysis was a subject to which the Hindu mind showed a happy adaptation ... the glory of having invented general methods in this most subtle branch of mathematics belongs to the Indians."

These equations are also called Diophantine equations in the honour of Diophantus of Alexandria (c.250 CE), the adjective "Diophantine" pertains not so much to the nature of the equation as to the nature of the admissible solutions of the equation.

Diophantus actually investigated solutions in *rational* numbers, a topic of considerable geometric importance. But finding the solutions in *integers* is a much harder problem. As

L.E. Dickson explains in the Preface of his famous text "History of the Theory of Numbers (Vol II)":

> "...it is a real difficulty to pick out those points of the conic whose coordinates are rational and a greater difficulty to pick out those points whose coordinates are integral."

Further, unlike Diophantus who was satisfied with *one* particular rational solution, the Indians investigated *all* possible integer solutions to the problem.

> "The object of the former [Indian] was to find all possible integral solutions. Greek analysis, on the other hand, demanded not necessarily integral, but simply rational answers. Diophantus was content with a single solution; the Hindus endeavored to find all solutions possible." ([Ca, p 95])

Indian research works in this area of mathematics were too far ahead of the times to be noticed by contemporary and subsequent civilisations. As Cajori laments [Ca, p 97–98]:

> "Unfortunately, some of the most brilliant results in indeterminate analysis, found in the Hindu works, reached Europe too late to exert the influence they would have exerted, had they come two or three centuries earlier."

The linear indeterminate equation $ay - bx = \pm c$ (a, b, c positive integers) has infinitely many integer solutions when the greatest common divisor of a and b divides c (otherwise the equation does not have any integer solution). This fact was known to ancient Indian mathematicians and astronomers. They evolved an efficient method called *kuṭṭaka* (pulverisation) for finding the solutions and applied the method to solve problems in astronomy. The earliest description of the *kuṭṭaka*

occurs in a highly condensed form in the Āryabhaṭīya (499 CE) of Āryabhaṭa.

Brahmagupta brought clarity and refinement in the presentation of Aryabhata's algorithm for finding integer solutions to the linear indeterminate equation $ay - bx = c$.

Solutions of Quadratic Indeterminate Equations

Brahmagupta took up the harder problem of investigating the quadratic indeterminate equation $Dx^2 + m = y^2$, where D is a positive integer which is not a perfect square. The equation was called *varga-prakṛti* (square-natured). Special emphasis was laid on solving the important case $Dx^2 + 1 = y^2$.

Note that if D is negative, or if D is a square of a positive integer, then $Dx^2 + m = y^2$ has only finitely many integer solutions. In fact, it is easy to see that $(\pm 1, 0); (0, \pm 1)$ are the only solutions of $Dx^2 + 1 = y^2$ when $D = -1$; and that $(0, \pm 1)$ are the only solutions of $Dx^2 + 1 = y^2$ when D is a negative integer less than -1 or when D is a positive integer which is a perfect square. Thus the problem becomes mathematically interesting only when D is a positive integer which is not a perfect square. In this case we now know that the equation $Dx^2 + 1 = y^2$ has infinitely many integer solutions. The general equation $Dx^2 + m = y^2$ need not have any integral solution. For instance, $3x^2 + 2 = y^2$ does not have any integer solution. However, when it does have an integral solution, it has infinitely many.

Law of Composition

The first major breakthrough on the problem of solving $Dx^2 + 1 = y^2$ in integers was achieved by Brahmagupta through a brilliant innovation — a law of composition called *samāsabhāvanā*. In modern language and notation, his principle can be formulated as follows:

Theorem 1. [Brahmagupta's Bhāvanā]
For a fixed positive integer D, the solution space of the equation $Dx^2 + m = y^2$ admits the binary operation

$$(x_1, y_1, m_1) \odot (x_2, y_2, m_2) = (x_1y_2 + x_2y_1, Dx_1x_2 + y_1y_2, m_1m_2).$$

In other words, if (x_1, y_1, m_1) and (x_2, y_2, m_2) are solutions of $Dx^2 + m = y^2$, then so is $(x_1y_2 + x_2y_1, Dx_1x_2 + y_1y_2, m_1m_2)$.

This composition principle of Brahmagupta is of paramount significance in modern algebra and number theory. When Euler rediscovered it, he called it a *theorema eximium* (a theorem of capital importance). It is a statement of the multiplicativity of a "norm function", a very important concept in modern mathematics. It is also a statement on the composition of binary quadratic forms, another important and rich topic which is still an active area of research. The very idea of constructing a binary composition on an abstractly defined unknown set is the quintessence of modern "abstract algebra". This single work, at such an early date, is enough to mark him as one of the greatest algebraists in the history of mathematics. The discovery of algebraic structures, like Theorem , on a set of significance, is now an important theme in mathematics research. For further discussion, see [D1, p 81–89] and [D2, p 180–183].

Applications of the Law

After stating his composition law (Theorem 1), Brahmagupta immediately derived infinitely many rational solutions of $Dx^2 + 1 = y^2$. He then used the composition rule to generate an infinite number of integer solutions from a given integer solution of $Dx^2 + 1 = y^2$. This result was applied later by Nārāyaṇa (c.1350) to generate a sequence of progressively better rational approximations of $\sqrt{D}$. Note

that, if $Da^2 + 1 = b^2$, then

$$\frac{b}{a} - \sqrt{D} = \frac{b^2 - Da^2}{a(b + a\sqrt{D})} = \frac{1}{a(b + a\sqrt{D})}$$

and thus, for a sufficiently large solution (a, b), $\frac{b}{a}$ will be a good approximation for $\sqrt{D}$; the larger the numbers a and b, the better the approximation.

More generally, when an equation $Dx^2 + m = y^2$ has an integral solution, Brahmagupta used that solution and a non-trivial integral solution of $Dx^2 + 1 = y^2$ to generate infinitely many integral solutions of $Dx^2 + m = y^2$.

Brahmagupta's methods enable one to find integer solutions of $Dx^2 + 1 = y^2$ in a wide variety of cases. Brahmagupta mentions two illustrative difficult equations $83x^2 + 1 = y^2$ and $92x^2 + 1 = y^2$ where his methods apply. If one has an integer solution to any of the equations $Dx^2 + m = y^2$, where $m \in \{-1, \pm 2, \pm 4\}$, one can derive a positive integer solution of $Dx^2 + 1 = y^2$ by repeated use of Theorem 1.

While Brahmagupta had given a partial solution to the problem, another Indian algebraist subsequently discovered the complete integer solution by a cyclic method called *cakravāla* (*cakra*: wheel or disc) in ancient India. Brahmagupta's law is an important component of the *cakravāla*. Quite often, it also provides a rapid short-cut after a few steps of the *cakravāla*.

A more elaborate description of Brahmagupta's results and applications on this topic is given in [D1, p 77–79; 90–99] and [D2, p 176–186].

The equation $Dx^2 + 1 = y^2$ in Europe

The problem of solving the equation $Dx^2 + 1 = y^2$ in integers was raised in Europe in the 17th century by

Pierre de Fermat (1601–65) as part of his effort to create enthusiasm for number theory (the study of *integers*) among his contemporary mathematicians. Fermat was sure that mathematicians who can successfully apply themselves to this problem will appreciate the beauty and intricacy of number theory. A general method for solving Fermat's problem was discovered by Brouncker in 1657. During 1657–58, there had been an exchange of letters among mathematicians like Frenicle, Brouncker and Wallis who had taken interest in Fermat's problem. In 1685, Wallis published his monumental work on algebra in 100 chapters; chapter 98 was devoted to the problem of solving $Dx^2 + 1 = y^2$. The problem was again taken up in the next century by the two greatest figures of 18th century mathematics: L. Euler (1707–83) and J.L. Lagrange (1736–1813). Euler became interested in the problem from around 1730 and Lagrange took it up around 1768. They developed the theory of continued fractions and investigated the problem in the framework of this theory. Due to an error, the equation $Dx^2+1 = y^2$ came to be known as Pell's equation.

Importance of the equation $Dx^2 + 1 = y^2$

The study of indeterminate equations, especially the study of the so-called "Pell's equation", played an important role in the evolution of classical algebra.

In modern algebra, the solutions of the equation $Dx^2+1 = y^2$ yield units in the domain of integers of the quadratic field $\mathbb{Q}(\sqrt{D})$. The equation is also closely related to the study of binary quadratic forms. The solution of this equation is the main step in the solution of the general quadratic Diophantine equation in two variables. It also played a role in the solution (1970) of Hilbert's 10th problem on the non-existence of an algorithm for solving arbitrary Diophantine equations.

Pell's equation continues to fascinate mathematicians even today. Research papers and articles are being published on it in various contexts. Efficient generation of solutions of the equation is a very active area of research in algorithmic number theory and computer science.

More historical details are given in [D2, p 149–155].

Epilogue

Among Brahmgupta's many achievements, it is in the treatment of the equation $Dx^2 + 1 = y^2$ that we can best fathom the depth and profundity of his algebraic mind. It is amazing that the person who crystallized algebra as a distinct discipline, and had to organise its foundations, had further ascended to sophisticated layers of abstract algebraic thinking, a feat that must rank among the most magnificent triumphs of the human mind. To comprehend his greatness, we have only to consider some aspects of the general evolution of algebraic thought and recall the time taken in modern Europe for the emergence and flowering of the entire range of ideas present in Brahmagupta's work. Till the 16th century, Arab and European mathematicians had struggled even with problems involving equations of the type $ax + b = c$. While negative numbers were accepted and symbolism in algebra was perfected in Europe in the 16th century, Fermat discussed the equation $Dx^2 + 1 = y^2$ in the 17th century, Euler rediscovered Brahmagupta's work in connection with this problem in the 18th century, and abstract algebraic ideas germinated in Euler and Lagrange in the late 18th century before coming out into the open in the 19th century.

The legacy of Brahmagupta reminds one of the following passage from Sri Aurobindo ([A, p 185]).

> "In what field indeed has not India attempted, achieved, created, and in all on a large scale and yet with much attention to completeness of detail? Of her spiritual and philosophic achievement there can be no real question.... But if her philosophies, her religious disciplines, her long list of great spiritual personalities, thinkers, founders, saints are her greatest glory, as was natural to her temperament and governing idea, they are by no means her sole glories, nor are the others dwarfed by their eminence. It is now proved that in science she went farther than any country before the modern era, and even Europe owes the beginning of her physical science to India as much as to Greece, although not directly but through the medium of the Arabs.... Especially in mathematics, astronomy and chemistry, the chief elements of ancient science, she discovered and formulated much and well and anticipated by force of reasoning or experiment some of the scientific ideas and discoveries which Europe first arrived at much later, but was able to base more firmly by her new and completer method."

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KNOWING THE WORLD BETTER THROUGH MATHEMATICS:
Bringing together 'Critical Mathematics Education' and 'Everyday Mathematics'

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Introduction

I rise to speak today to discuss as well as gather from your esteemed experience some of the concerns that we as educators and researchers face in our contemporary society with regard to the social aspect of mathematics learning and teaching. This aspect has remained a neglected area in mathematics education research (Bishop, 1988a) though of late, research in the socio-cultural aspects of mathematics education has picked up. I would like to focus on the interface of ethnomathematics and critical mathematics education and their possible bearings on mathematics teaching and learning. The approach that I plan to explore here today goes in consonance with the fact that mathematics learning is a cultural and social process and that this approach would allow us to critically analyse what passes for mathematics education at present. I start with an instance that I encountered during one of my field-trips to a Mushari (hamlets where Mushars live) in Bihar in the winter of 2008-'09.

Once during an interaction with Mushars, one of them had the following comment and question:

[*]Professor A.Narasinga Rao Memorial Lecture delivered by Prof. Arindam Bose, at the 45 th Annual Conference of AMTI held at Kolkata, during December 27-29, 2010.

"I have taught my son the kind of hisaab (arithmetic) he's going to need in life..."

"What are our children going to learn in school? Why do we send them to school?"

Mushar (meaning 'rodent-hunters') is a socially marginalised community of people that dwells in different parts of north India. They are considered a part of the Mahadalit, or the "untouchables" and come under the category "scheduled caste". As a common practice, people from this subaltern community face social boycott and live in groups generally on the outskirts of villages and do not have much access to formal education system. Without any land or regular work or employment they live in abject poverty and earn their livelihood by working as labourers or selling country-made liquors. In the backdrop of the above questions that Mushars raised, it becomes all the more important for us, the mathematics educators, to find the meaning of education, mathematics education in particular, for them and to make sense of the perception that prevails among a sizeable section of our society about mathematics learning. More so, in the light of the inclusiveness of school education that people at the helm of affairs in our country are worried about these days. The complex relationships between education and differential power, the issues of social justice and equity come to the fore only when we discuss such 'frustration' (as reflected from the above questions) that has remained subdued in a large chunk of our population over decades. It raises the question as to whose knowledge do we deem as 'legitimate' so as to be taught in schools, become part of the curricula and fetch the tag of being 'formal'. What all criteria are used to decide as to what must go into the curricula and what do we want our children to learn?

The answer to many such questions might come from the works of critical mathematics educators. One of them, for example, Eric Gutstein (2006) quotes Delpit (2008) to assert that, "we must keep the perspective that people are experts on their own lives... they can be the only authentic chroniclers of their own experience". Community-knowledge for example, not only is significant and workable but is shared among the community members too. Therefore, educating our children mathematically requires much more than mere teaching of mathematical facts. It calls for the connection between what Alan Bishop (1988a, 1988b) calls the values that underlie mathematics, the complexities that are involved in educating our children about those values and the community-knowledge, apprenticeship and mathematical facts.

Many mathematics educators, especially the ones who are working towards a pedagogy towards critical mathematics education like Ole Skovsmose, Brian Greer, Eric Gutstein, Bill Atweh remind us that mathematics teaching can be used as a tool to "help make the world more equal and just" (Gutstein and Peterson, 2006) and this approach of weaving social justice issues throughout mathematics curriculum would "deepen students' understanding of society and to prepare them to be critical, active participants in a democracy" (*ibid*, p.1) and make them learn about the "lives of people in various parts of the world and the relationship between the things we consume and their living conditions". This way mathematics happens to be an essential tool for understanding and changing the world as well (*ibid*).

What is Critical Mathematics Education?

Critical mathematics education (CME) brings forth the concerns related to mathematics and its role in society. It serves an emancipatory interest that aims at bringing about

a liberatory social change with a guided activity through critiques. CME is not expected to take any prototypical format and therefore there cannot be any specified or certain approaches with general applicability (Alro, Ravn, Valero, 2010). It is an approach to mathematics education that can lead to both critical consciousness as well as social justice. Critical education in general involves problem posing for all the stakeholders requiring them to transform their actions based on reality. CME helps in understanding "how to use mathematical ideas in struggles to make the world better?" or, to be specific, the key questions are: "Do the *real* real-life mathematical word problems make the social justice issues more clear?" and, "Does the clarity lead to actions for social justice?" (Frankenstein, 2010).

Mathematics as a School Subject

Mathematics is an important school subject taught right from the kindergarten level across most of the contemporary cultures. However, as Bishop (1988a) asserts, it is one of the least understood subjects and not many people feel comfortable with it. It is often taught and perceived as a subject that is completely detached from the real world. The major policy document that is currently followed in our country, the *National Curriculum Framework* (NCF, 2005) points out that "the teaching of mathematics should enhance the child's resources to think and reason, to visualise and handle abstractions, to formulate and solve problems" (p. ix). It further emphasises that the subject matter of mathematics can be understood better if the teaching is embedded in the child's experiences. The NCF points out that "learning takes place both within school and outside school" and that "learning is enriched if the two arenas interact with each other" (p. 15). The Framework gives great importance to connecting school learning with the child's lived experience, "not only because

the local environment and the child's own experiences are the best entry points into the study of disciplines of knowledge, but more so because the aim of knowledge is to connect with the world" (p. 30). Connecting with the child's environment also has a role to play in creating an educational culture that is equitable. It is important that our children feel that "each one of them, their homes, communities, languages and cultures, are valuable as resources for experience to be analysed and enquired into at school; that their diverse capabilities are accepted" (p. 14). The position paper of the Focus Group on the teaching of Mathematics expresses the same concern by emphasising the use of 'experience and prior knowledge' to construct new knowledge in school mathematics (Position Paper, Mathematics, 2005, p. 8).

However, the current model of teaching and learning of mathematics that is actually followed in schools places more emphasis on the absorption of already existing ideas (in the textbooks) by the students with little or no scope for them to draw upon from their own informal or everyday conception of mathematical ideas. While the authority and expertise lie completely with the textbooks and the teacher, mathematical ideas generated from the out-of-school contexts and practices are not acknowledged as relevant because such knowledge may not conform with what is regarded as 'genuine' mathematical knowledge according to the textbook and teacher's interpretation (Millroy, 1992). Post-Piagetian constructivism has emphasised the fact that a child's mind is not a *'tabula rasa'*, that children studying in elementary classes enter their schools with prior knowledge drawn from their environment and everyday experiences. Hence, it is of importance to a community of mathematics educators to investigate what all children know about mathematics drawn

from the outside world and how this impacts the learning of school mathematics.

What is 'Everyday Mathematics'?

'Everyday mathematics' refers to the form of mathematics that people make use of in out-of-school settings while engaging in contextually embedded practices. Most of the studies done in 'everyday mathematics' (terms like 'informal mathematics' or 'street mathematics' or 'out-of-school mathematics' are also used) indicate that "when children learn a numeration system and understand it well they can invent ways of using it to solve arithmetic problems through counting and decompositions" (Nunes, Schliemann and Carraher, 1993). All these studies indicate that it is one thing to learn mathematics in schools in a formal way and quite different to solve contextualised mathematical problems in everyday activities (*ibid*). 'Everyday mathematics' has different forms based on the goals and activities it is intertwined in. Knowledge of everyday mathematics has not remained confined with arithmetical problem solving alone but there are wide range of activities that use such knowledge, for example, measurement and estimation of length, volume and time; geometry; logical reasoning; tachistoscopic estimation of numbers; etc.

Studies carried out in Liberia in the mid-seventies (Lave, 1988), in Papua New Guinea in the late seventies and in the early nineties (Saxe, 2002, 1992, 1988), in Brazil in the eighties and nineties (Carraher et al., 1987; Nunes et al., 1993, 1985) have looked into the different aspects of 'everyday mathematics' (or street mathematics or informal mathematics) developed in out-of-school settings and their comparisons with school mathematics. Most of the studies indicate that participants who were untrained in school mathematics could

competently perform the calculations needed in their workplace activities. In contrast, the school students, trained in school mathematics when presented with such problems came up with incorrect solutions or even absurd solutions. School students concentrated more on the numbers given in the problems and paid little attention to the meanings of the problems. (Nunes, Schliemann, and Carraher, 1993, 1985; Lave, 1988). On the other hand, street vendors who with 'impressive ease' solved their routine problems in everyday settings, could not solve the same types of problems which they had earlier solved in their workplace contexts when presented to them as formal word problems without any context. Sometimes they gave insensible solutions, for example, getting as an answer a number in a subtraction problem that is bigger than the minuend (Nunes, Schliemann and Carraher, 1993).

Distinction between 'Everyday Mathematics' and 'School Mathematics'

Nunes, Schliemann and Carraher (1993) and Lave (1988) have shown through their empirical studies that solving problems in out-of-the-school settings intertwined in everyday activities is quite different from the formal ways of solving them using school taught techniques. Out-of-the-school contexts for mathematics include shopping, household activities, games, workplaces or various other places that involve informal daily-life mathematical practices. In everyday mathematics the doer has a continuous engagement with the objects and the situations and she does not burden herself with the extra effort to remember the algorithms. calculation-techniques and the reasoning used (Resnick, 1987). This is in contrast to school mathematics where one does not usually have a freedom of making a choice of using alternate techniques other than

those taught in the classrooms. In schools, mathematical activities are based on symbols which get detached from any meaningful context. Rather more stress is usually given to symbol manipulations and following rules (*ibid*). Drill and practice have remained a major means of instruction in elementary arithmetic classrooms (Resnick and Ford, 1981). Though this is done with the aim of developing computational proficiency including developing speed and accuracy, following the correct steps in the algorithms gets more importance than their relevance and understanding. School mathematics is aimed at improving individuals' performances and skills, whereas, out-of-school mathematical activities are socially shared. While school mathematics focuses on generalised learning, everyday mathematical ability grows from situation-specific competencies (*ibid*).

'Everyday Mathematics' through Everyday Cognition

A glimpse of 'everyday mathematics' came up during my interaction with the Mushars which indicate how knowledge drawn from everyday activities and experience influence the calculations that they make. For example, the following interaction with a Mushar (60 years, Male, non-literate) who was cutting the roots of an already felled tree, shows a similar result: (AB: interviewer, NM: interviewee, RP: key informant, 1 maun = 40 kg approx.).

AB: Suppose you sell wood at Rs.16 per maun, how much do you get for 4 maun?

NM: (with a pause) Rs. 64.

AB: How did you work that out?

NM: How can I say that? It just comes to my mind.

RP: Think a little and tell us. Suppose I pay you Rs 60 only, then there will be a fight.

NM: No, no, why should there be a fight on this? I will accept it. I don't fight.

RP: Still, think for a while.

AB: What went on in your mind? What brought you to 64?

NM: umm..15 doubles gives 30, doubled once more gives 60 and another 4 gives 64.

AB: Now, if the price were Rs 17 per maun, how much do you get?

NM: That requires more thinking...

This indicates that people tend to use the numbers that they are convenient in working with. For example, 60 is a convenient number for NM. This becomes even more clearer from the following interaction with another Mushar VM. VM had informed AB that one bottle of liquor is sold for Rs 25. The interaction went like this:

AB: How much will you sell 30 bottles of liquor for?

VM: (after a pause) Rs 750.

AB: How did you do that?

VM: 4 bottles fetch Rs 100, 8 bottles Rs 200, 12 bottles Rs 300.

VM: (after a pause) so, Rs 700.

Now, 2 more bottles are to be counted, that's Rs 50. Hence, Rs 750.

He later explained that in Rs 700 come 28 bottles.

Therefore, altogether for 30 bottles, price of two more bottles needs to be added which makes it Rs 750.

This calculation-strategy gives a notion of parallel counting as well. VM was calculating two things simultaneously. Every time he added 100, he kept track of the numbers of bottles too! There are many such examples of 'double counting' or 'parallel counting' that people like VM ought to do while doing trades. He complained that while trading many try to capitalise on their lack of arithmetical knowledge. This prompts him to remember few numerical

facts which he had observed while trading. This realisation of the importance of knowledge of mathematics gives rise to the issues of social justice and equity that a pedagogy in CME can address.

There was another interaction with a farmer (non-Mushar), DH (50 years, male, non-literate), who calculated 35×10 when given a contextual problem for 35×10 (price of wheat is Rs 35 per kilo, then what is the price of 10 kilos of wheat?):

DH: 3 times 35 give 105 and one more 105 gives 210. So, umm.. 6 over.. Left with 4. Take one more 105, and that's 315. Now that's 9 over.. So, add one more 35. So, that's 350. That's full 10. Yes, yes, the answer should be 350.

This strategy of calculating 'ten times' or 'multiplication by 10' is identical with Terezinha Nunes' finding with Brazil's street vendors (coconut vendors) (Nunes, Carraher and Schliemann, 1985). She has reported similar account of calculations. What is striking here is the fact that unlike the prescription in academic mathematics that one 'zero' needs to be juxtaposed to the multiplicand from the right when multiplied by ten, people use their 'closure' strategies and use their 'convenient numbers' while working with 'everyday mathematics'. Such accounts may not be recognisable through our eyes as Millroy (1992) says, which are "shaped by many years of studying and teaching formal academic mathematics".

When asked why he did not consider 35×2, DH replied. "It will be difficult as then, I'll have to remember many numbers". However, another Mushar RM (25 years, male, non-literate) attempted at doubling 35 first and eventually arrived at the answer successfully. He explained in the following way:

RM: Double of 35 is 70 and doubling again given 140. That covers 4. So, five is 175 and doubling that gives 350.

This indicates that people tend to narrow down upon a strategy as per their convenience based on their practice and experiences. One can also argue here that they try to move towards 'closed spaces' and that they have a sense of 'closures', for example, moving towards 20, 50, 100, etc which gives a sense of 'completing' a 'set'.

Culturally-embedded Mathematical Practices

In addition to the 'everyday' strategies, different cultural groups implicitly hold different practices including mathematical practices that might be argued as culturally embedded. Study of such practices gives rise to a different branch within the domain of mathematics education that is referred to as 'ethnomathematics'. Ethnomathematics is a collection of ideas and notions about the ways mathematics is perceived and used by different cultural groups and how they do mathematise viz. counting, measuring, relating, symbolising, classifying and inferring (D' Ambrosio, 2006). It refers to the cultural roots of mathematics and the implicit mathematics that people engage in, in everyday settings. As an educational domain 'ethnomathematics' strives to unravel and comprehend the codes that a cultural group makes use of in order to describe and understand the real world around it. It seeks to trace the distinguishable practices of the identifiable cultural groups employed through the domains of cognitive and cultural anthropology. It is a way to identify the origin of these cultural practices and their diversities. Ethnomathematics brings forth different types of mathematics that have come up from the interactions of humankind with the environment. People within their communities develop a common system for understanding and dealing with various quantities and

their measurements; and relational and spatial understanding, drawn from their respective cultures. Ethnomathematics looks into such practices and explores the 'instances of diverse expressions of mathematical ideas' (Ascher, 1991). Barton (1996) suggests that ethnomathematics is all about 'making sense'. He defines it as, "Ethnomathematics is a research programme of the way in which cultural groups understand, articulate and use the concepts and practices which we describe as mathematical, whether or not the cultural group has a concept of mathematics". Sometimes people of a cultural group might not be aware of the fact that their practices involve mathematics and they do not have the concepts in formal sense. Ethnomathematics also involves the deliberations of issues related to society and culture and their impact on teaching and learning of mathematics.

The meanings attributed to the term 'ethnomathematics' keep modifying. Different practices have different ethnomathematics based on different cultural groups' mathematical practices and processes of knowledge-making. We can see the different practices involving mathematics in the works of Paulus Gerdes (in Mozambique), Marcia Ascher (in America), D'Ambrosio, David Carraher and Terezinha Nunes (in Brazil), Geoffrey Saxe (in Papua New Guinea).

During the interaction with the Mushars, I came across few mathematical riddles (called 'Kuttaka') that are part of their recreational mathematics and folk mathematics. Riddles are part of the folk mathematics in many cultures. It has been suggested that riddles are important for literacy and numeracy education (Rampal, 1998). Some of the riddles are reproduced below:

take take haari,
paanch take bhaari,

take ke bees go tuiyaan,
sau saaman sau rupaiya.

[An object (haari) comes for one rupee, another comes for 5, 20 of still another comes for a rupee, 100 rupees for 100 objects]

10 takia, 5 takia, ek takia,
Teeno milake rupaiya 480,
Teeno hain barabar,
Kai-kai saaman, kai-kai- takia?

[Rs 10, Rs 5, Re 1, 3 items cost Rs 480, numbers of items are equal, how many items for how much each?]

Teen janwar hain,
gai, bhains aur bakri,
gai deti aadha kilo doodh roj,
bhains deti 4 kilo, bakri pauua kilo roj,
20 go janwar chahi auu 20 kilo dhoodh,
Kai-kai janwar - gai, bhains, bakri?

[There are 3 animals - cows, buffaloes and goats, cow gives half kilo milk everyday, buffalo 4 kilos and goat quarter kilo everyday, 20 animals needed & 20 kilos of milk, how many each of them would you take?]

titir ke do aage titir,
titir ke do pachhe titir,
titir titir kai titir?

[a bird has two birds in the front, a bird has two birds behind, Tell tell how many birds?]

Several of the riddles are based on indeterminate equations with positive integer solutions. Often constraints are put

which make them determinate equations. The solutions of determinate equations have played an important part of the historical development of algebra in India. The study with Mushars indicated the integration of their arithmetical knowledge acquired from daily life practices and knowledge of riddles disseminated through generations in their culture and that this may have been part of the oral practice since the development of algebraic equations in India (Bose, 2009). Some of the members of the Mushar community were not only able to solve such riddles but were able to pose more. The researcher re-formulated on such riddle (changing few numbers) and posed to them which they could solve as well. All this with no background in formal education or schooling.

The re-formulated riddle asked to them was:

> *20 takia, 10 takia, 2 takia,*
> *Teeno milake rupaiya 480,*
> *Teeno hain barabar,*
> *Kai-kai saaman, kai-kai takia.*
>
> [Rs 20, Rs 10, Rs 2, 3 items cost Rs 480, no of items are equal, how many items for how much each]

The Problem

The International Association for Educational Achievement in its analysis of the mathematics curricula of 22 countries the world over reports that there is "substantial common core in most areas of mathematical content across curricula" (Millroy, 1992). But the irony is that Mathematics remains an abstruse subject for many. Mathematics often acts as a filter. it filters students out of the school. So much is the fear for the subject that unpleasant childhood experiences of the mathematics classrooms often persists for long, even

later in lives. Therefore, the problem of teaching and learning of mathematics is universal in nature. There seems to be a two-fold problem: one that of accessibility and the other that of relevance and meaningfulness. The former raises the social justice and equity issues while the latter is about making connections between the formal academic mathematics with the lived experience, i.e. 'everyday mathematics'.

Final Comments

Everyday mathematics and mathematical practices that are culturally embedded are phenomena of social significance. Their role is important even if official recognition eludes such mathematical abilities that are actually needed in real-life situations, every day and everywhere. 'Everyday mathematics' is indeed an example of cognition in practice as Jean Lave calls it (Lave, 1988). We must realise that "mathematics is a tool for understanding a situation" (Nunes, Schliemann and Carraher,1993). In fact, one of the possible lessons that we can draw upon from 'everyday mathematics' and take to our classrooms could be 'meaning preserving' that happens when we mathematise a real-life situation. This concept was deeply explored under 'Realistic Mathematics Education' project that was developed by the researchers of the Freudenthal Institute, Utrecht. It focuses on the empirical constraints and 'common understanding' that are made use of in out-of-school practices. Classroom teaching need not be about partitioning between mathematical knowledge and its application, rather it ought to be the other way round, i.e. applications should lead to factual development of mathematical concepts and skills. Students have their own representations of the situations, mathematics can provide them tools that can help them connect several aspects of the reality with their (pre)-knowledge, i.e. using their own resources to make sense of the problem (read: 'world'). This allows children to make co-ordinations between

different (pragmatic) schemas and forms of representations that they learn from 'everyday mathematics'. The challenge for the educators is to "convert such potential for learning into conscious routes for teaching" (*ibid*).

The Critical Mathematics Education on the other hand, allows examination of the various decisions that are taken based upon 'assumed' mathematical models that are part of our everyday lives (for example, various government policies). Such decisions generally have serious social ramifications. The questions that Mushars raised are outcomes of the grave social consequences of different political, educational and social decisions taken over the decades. Mathematics education helps in addressing such issues. It is essential for the development of critical thinking that builds the power of questioning the use of different mathematical models and their effects on the lives of the people.

Across the globe, Mathematics learning and teaching happens in the contexts of linguistic and cultural diversity. How is this diversity negotiated, how do we work with and within this diversity for enhancing the understanding of mathematics is the moot question. The formalists often wish to set an international standard in school mathematics achievement, but such uniformity stands the chance of undervaluing and ignoring the enrichment and relevance of the cultural practices and community knowledge that individual curricula may bring forth. Our main objective of facilitating successful mathematics learning can therefore be attained with the recognition of curricula that is developed with the integration of pedagogy of social justice and the culturally embedded practices. I understand that this link will provide a natural connection between knowledge drawn from 'everyday mathematics' and issues arisen from critical mathematics education. Such a connection can help us know ourselves and our world better.

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A NOTE ON DIGITAL ROOTS

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Let n be a positive integer. Then find the sum of the digits of n. If the resulting number is not a single digit number, then repeat the above process. We continue this process until a single digit number is obtained. This single digit number is called the digital root of n. That is, the ultimate single digit number one gets by applying digit sum (repeatedly, if necessary) is the digital root of n. For example, consider 679. The sum of its digits is $6+7+9=22$. The sum of the digits of 22 is $2+2=4$, a single digit number, which is the digital root of 679. A single digit number is its own digital root.

Clearly, the digital root of any positive integer is one of $1,2,...,9$. If a and b are positive integers, then it is obvious that digital root of $(a+b)$ = digital root of (digital root of a + digital root of b), and digital root of (ab) = digital root of (digital root of a multiplied by digital root of b).

Let us denote the digital root of the product ab by $a*b$. For example, $12*8 = 6$. It is clear from the following composition table that $G = \{1,2,4,5,7,8\}$ is a cyclic group with respect to $*$.

	1	2	4	5	7	8
1	1	2	4	5	7	8
2	2	4	8	1	5	7
4	4	8	7	2	1	5
5	5	1	2	7	8	4
7	7	5	1	8	4	2
8	8	7	5	4	2	1

From the table we find that G is closed with respect to $*$;

$*$ is both associative and commutative; identity element is 1; inverses of $1, 2, 4, 5, 7, 8$ are respectively $1, 5, 7, 2, 4, 8$.

Now $2^2 = 2 * 2 = 4, 2^3 = 4 * 2 = 8, 2^4 = 7, 2^5 = 5, 2^6 = 1$. Hence $G = \{2, 2^2, 2^3, 2^4, 2^5, 2^6\}$, so that G is a cyclic group of order 6 with respect to $*$.

Note that 5 is also a generator of G. The proper subgroups of G are $\{1\}, \{1, 4, 7\}$ and $\{1, 8\}$.

Recall that there are only two types of groups of order 6: one is cyclic and the other is the non-commutative group isomorphic to the group S_3 of permutations on $S = \{1, 2, 3\}$.

If we define $a \oplus b =$ digital root of $a+b$, then as above it is easy to see that the set $\{1, 2, 3, 4, ..., 9\}$ is an abelian group with respect to $\oplus$. Identity element is 9; inverse of $r (1 \leq r < 9)$ is $9 - r$ and inverse of 9 is 9. $\{3.6.9\}$ is a subgroup of H.

Now, for each $r (1 \leq r \leq 9)$, define $C_r = \{m | m$ is a positive integer with digital root $r\}$ and for r, $s (1 \leq r, s \leq 9)$, define $C_r \oplus C_s = C_t$, where t is the digital root of $r + s$; and $C_r * C_s = C_l$, where l is the digital root of rs. Then it is easy to see that $\{C_1, C_2, C_4, C_5, C_7, C_8\}$ is a group with respect to $*$ and $\{C_1, C_2, \cdots C_9\}$ is a group with respect to $\oplus$.

PARADOX, TRUTH, AND COMPUTING

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1. The Part of Philosophers

1.1. The Liar. It's a strange place to start a talk about the mathematics of computation, but we do start on the Mediterranean island of Crete, about 600BC, with a religious dispute. Epimenides, a holy man in that country, came into dispute with the prevailing religion in the area over whether or not the god Zeus was mortal. Epimenides wrote,

> They fashioned a tomb for thee, O holy and high one—
> The Cretans, always liars, evil beasts, idle bellies!
> But thou art not dead: thou livest and abidest forever,
> For in thee we live and move and have our being.[1]

The trick is this: Epimenides himself is a Cretan. So when he says, "The Cretans, always liars...," it gives us something to

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[1] J. Rendel Harris, "Note on the Cretans," *The Expositor* 2 (1906) pp. 305–317

ponder. If Cretans are *always* liars, then everything they say must be false. Including this. This trick isn't so hard. Maybe it's just not true that Cretans are always liars.

But there is a harder variant. About two hundred years later, Eubulides, a Greek,[2] posed this problem:

> What I am now saying is a lie.

Now, there's really no getting around this one. If it's true, then it's false. If it's false, then it's true.

The old-time Muslim philosophers loved this. They had a debate of their own in progress. The grammarians like to categorize things, and one of the first things they categorize is sentences. But even if we know a declarative sentence when we see one, it's not easy to give a general prescription for one. So there was one possible definition saying that a declarative sentence was something that could have a truth value — it could be true or false. "What time is it?" can't be true or false. But "It's 3:07" can. This "Liar" sentence became a favorite example, since it's clearly a declarative sentence, but can't have a truth value.

By the thirteenth century, though, there were Islamic treatments of the liar paradox for itself. The astronomer Tusi explained the problem in this way:

> If a declarative sentence, by its nature, can declare something about anything, then it is possible that it itself can declare something about another declarative sentence.

[2]Diogenes Laërtius, *Lives of the Eminent Philosophers*, book II, section 108.

So you can have the sentences

1. Sentence 2 is false.
2. The professor is sitting.

But as soon as we have this, we can have something like

1. Sentence 2 is false.
2. Sentence 1 is true.

From here, it's not hard to think of the sentence, "What I am saying now is a lie," as playing both of these roles: first, as a sentence saying something about a subject, and second as the subject being talked about.

Tusi's insight is that the liar paradox isn't nearly as bad as a true cynic would like to say. I mean, a true cynic would like to say that because of this, truth is meaningless. But if you're trying to say that truth is meaningless, this is sort of a cheating way to do it. For truth to be meaningless, you should make it meaningless in an important case like, "My son was not well and had to leave school yesterday." The liar paradox strikes a false note to a practical person when it's used to talk about the nature of truth. So Tusi's solution was to say that you shouldn't cheat. Don't let sentences talk about themselves. Then you don't have silly problems like that. [3]

And this insight is important right down to the formal verification of the Airbus 380 avionics, which was done by Indian software engineers just in the last few years.

1.2 Check-mate. At this point, we can skip quite a lot of time, and find ourselves at the beginning of the 20th century

[3] A. Alwishah and D. Sanson, "The Early Arabic Liar: The liar paradox in the Islamic world from the mid-ninth to the mid-thirteenth centuries CE," *Vivarium* 47 (2009) 97–127.

in Europe, where a philosopher named Frege was trying to nail down exactly what people meant by saying that mathematical truths were "absolutely" true. This had been an old tradition in Western thought. Perhaps some things might be infected by political or religious prejudice. Maybe even our senses deceive us sometimes. But mathematics was thought to be completely and deeply true, in a way that other things are not.

There had been some obsession with absolute truth since the time of Descartes. He famously wrote that the things we think we know most surely could just be a very vivid dream. Or maybe there's some very powerful demon that deceives us. But mathematics was always special.

Now if you're going to give account of mathematical truths, you start with simple truths. It's *true* that every representation of a finite group has a unique decomposition into irreducible representations. But it takes a few years of university study in mathematics to even know what all of those words mean, and that's *not* the sort of thing whose truth was thought so incontrovertible.

Frege was looking to describe the arithmetic of natural numbers. Now one easy way to avoid Descartes' arguments is if mathematical truth doesn't depend at all on what we think. If it's completely outside our own minds, then whether we're dreaming it or not, whether there's a demon or not, it all comes out the same.

So the first thing he had to do was define numbers. That's not too hard, as philosophers reckon things. I've got a two-year-old son at home, and he absolutely loves idli. He can even, on a good day, identify, "Two idly." He can also (with equal enthusiasm) recognize "two chapati" or "two dosa." So the way Frege defined two was that it was the set of all objects of this kind. It's roundabout, but it works.

But then you have to say what "equal" means, and when your numbers are so abstract, it's not easy anymore. So Frege said that two sets are equal if any function you can define from them is the same on all arguments. Take a function from one, there's a function from the other where if you give each of them any input, you get the same output.

Comes a young guy at Cambridge — this was pre-Ramanujan Cambridge — Bertrand Russell. Being young, he wrote very humbly to the great Herr Professor Doktor Frege:

> I have known your *Grundgesetze der Arithmetik* for a year and a half, but only now have I been able to find the time for the thorough study I intend to devote to your writings. I find myself in full accord with you on all main points, especially in your rejection of any psychological element in logic and in the value you attach to a *Begriffsschrift* for the foundations of mathematics and of formal logic, which, incidentally, can hardly be distinguished. On many questions of detail, I find discussions, distinctions and definitions in your writings for which one looks in vain in other logicians. On functions in particular (Section 9 of your *Begriffsschrift*, I have been led independently to the same views even in detail. I have encountered a difficulty only on one point. You assert that a function could also constitute the indefinite element. This is what I used to believe, but this view now seems to me dubious because of the following contradiction: Let w be the predicate of being a predicate which cannot be predicated of itself. Can w be predicated of itself? From either answer follows its contradictory. We must therefore

> conclude that w is not a predicate. Likewise, there is no class (as a whole) of those classes which, as wholes, are not members of themselves. From this I conclude that under certain circumstances, a definable set does not form a whole. [4]

Frege wrote back,

> Dear Colleague,
>
> Your discovery of the contradiction has surprised me beyond words and, I should almost like to say, left me thunderstruck, because it has rocked the ground on which I meant to build arithmetic. It seems accordingly that the transformation of the generality of an equality into an equality of value-ranges is not always permissible, that my law V is false.... Your discovery is at any rate a very remarkable one, and it may perhaps lead to a great advance in logic, undesirable as it may seem at first sight.[5]

Years later, when Russell was old and distinguished himself, someone wrote him, asking permission to publish this correspondence in a collection of important historical documents. He wrote,

> I should be most pleased if you would publish the correspondence between Frege and myself, and

[4] B. Russell, Letter to G. Frege, translated and published in J. van Heijenoort, *From Frege to Gödel*, Harvard University Press, 1967, pp. 124–125

[5] G. Frege, Letter to B. Russell, translated and published in J. van Heijenoort, *From Frege to Gödel*, Harvard University Press, 1967, pp. 127–128

> I am grateful to you for suggesting this. As I think about acts of integrity and grace, I realize that there is nothing in my knowledge to compare with Frege's dedication to truth. His entire life's work was on the verge of completion, much of his work had been ignored to the benefit of men infinitely less capable, his second volume was about to be published, and upon finding that his fundamental assumption was in error, he responded with intellectual pleasure clearly submerging any feelings of personal disappointment. It was almost superhuman, and a telling indication of that of which men are capable if their dedication is to creative work and knowledge instead of cruder efforts to dominate and be known.[6]

So Russell made his own book, along with Alfred Whitehead (who also has some following as a religious teacher). In this *Principia Mathematica*, they carry out what Frege had attempted, without running into a contradiction. The key trick was exactly what Tusi recommended: Be much more picky about what sorts of things can be members of one another; about what sorts of things are allowed to be the subjects of sentences. But there was another problem.

2. The Part of Mathematicians

The German mathematician David Hilbert had spent the better part of a century going from one part of mathematics to another, settling all of its biggest problems, revolutionizing the way people looked at it. and then moving on to another field. By 1900, he was probably the most famous mathematician then

[6]B. Russell, Letter to J. van Heijenoort, published in J. van Heijenoort, *From Frege to Gödel*, Harvard University Press, 1967, p. 127

living. So when he gave a lecture on what he considered the most important problems for twentieth-century mathematics, it caught people's attention. Not all of the problems were like this, but a typical one was this:

> 10. Give a procedure to decide, for a polynomial $p(x_1, x_2, \ldots, x_n)$ with integer coefficients, whether $p(\bar{x}) = 0$ has any integer solutions. [7]

Always it was to give some procedure. And this did capture the spirit of the moment. If Russell and Whitehead and Frege were right, if one could lay out all mathematical truth in a formal system, then it seemed like one could just write down the problem, start calculating, and stop when an answer came up. Hilbert, who believed completely in this approach and had a rhetorical flair, said, "We must know; we will know."

Comes another young upstart, by the name of Kurt Gödel. Gödel, later in life, was a close friend of Einstein, but he was the friend that Einstein always had to look out for. Gödel had some rather severe mental health problems later in life, but even before they got very obvious, he was never what one might call normal. Anil Nerode, who is still living and who worked with Gödel in the late 1950's, says,

> What I remember that year about him is the two of us taking an Institute car every week of the winter into town. He wore a European heavy black overcoat and fedora. As it got to be a warm spring and a hot summer, I wondered when he would leave these at home. One day in June in blazing heat he had. He had changed over to a heavy European

[7] D. Hilbert, "Mathematical Problems," *Bulletin of the American Mathematical Society* 8 (1902), 437–479

> brown overcoat and brown fedora, which he wore the rest of the year. [8]

What Gödel did was this:

1. You can take any sequence of numbers and write it as a single number.
2. You can take any formal sentence and write it as a number.
3. You can write a proof as a sequence of sentences, and thus a sequence of numbers, i.e. a number.
4. You can then write down a sentence that says, "For any number encoding a proof of me, there is a smaller number encoding a proof of my negation."

This is Gödel's famous "incompleteness" argument. It doesn't say that anything in the *Principia* was wrong. It wasn't as devastating in that sense as Russell's paradox. But what it said was that any system of axioms that you can actually write down in any reasonable way — even if it's infinitely long — can't possibly be enough to decide all true statements about natural number arithmetic.

This means, first, that *Principia* isn't enough to carry out Hilbert's program, and second, that nothing else could ever do it, either!

Have you noticed that in a lot of great mythology — old myths and modern ones — important things show up in humble places? Well, in the midst of this abstract philosophical stuff, Gödel had just, unwittingly, given the first mathematically complete description of a computer.

[8] A. Nerode, "Autobiography: My Higher Education," `http://www.math.cornell.edu/~anil/highereducation.html`

3. Applications

The computing world has come a long way since Gödel, but there are some parts of this discussion that are still of extremely practical importance. The liar paradox (in the form of Gödel incompleteness) gives a limit on automated error-checking in software, and, at the same time, Russell's solution to the problem forms the basis for one of the best error-checking mechanisms we have, in the form of so-called "type-checking."

On another front, this paradox puts an interesting limit on artificial intelligence. The curious thing in each case is that we seem to know more standing outside the mathematics than the mathematics knows about itself. So if you want to build a computer that can think like a human, this is an important starting point: what is it that we can do that they can't, and how close can they come?

MATHEMATICS STOR(E)Y

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***Story** - telling is an art by itself. When you make use of it to explain concepts in Mathematics, it becomes a handsome tool for a teacher. An attempt is made here as an illustration:*

***Story 1: Grandma and** - 'Chakkarapongal':*

Once there lived a Grandma named *Raakayee* in a south Indian village. Her sons and daughters were settled in towns, cities and even abroad.

They invited her to their places, but *Raakayee* didn't oblige as she wanted to enjoy the rest of her life peacefully in her beautiful village.

The sons and daughters along with their children used to visit the village during the summer holidays every year to spend time with *Raakayee.*

In one of these summers, there were 17 grandchildren present at the village.

Their Grandma *Raakayee* prepared the south Indian sweet dish namely *Chakkarapongal* in 4 identical pots fully. She then served each of them equal portions.

Her grandchildren enjoyed feasting on the sweet dish and thanked their grandma for the same.

They also told her that they would be expecting the sweet dish for the next day too.

Their grandma told them that she will be preparing the sweet dish every day as long as they were present. *All of them jumped with joy.*

The very next day, 8 more grandchildren arrived at the place. Now, there were 25 grandchildren, more than the number of children the day before.

Raakayee prepared *Chakkarapongal* (again!) in 4 identical pots fully. She then served each of them equally.

Her grandchildren enjoyed feasting the sweet dish again and thanked their grandma for the same. One of her grandsons named Sunil was present on both these days.

a) On which day, Sunil would have got more *Chakkarapongal*?
[It should be the first day. Is n't it?]

b) Does this compare the two fractions $\frac{4}{17}$ and $\frac{4}{25}$?

Many with formal exposure to mathematics (or not) will be able to answer (a) but may not find it easy in (b).

The reason is quite simple! Most of them are able to connect the numerals with the characters unknowingly.

In this case, the numeral 4 plays the character' of 4 identical pots of *Chakkarapongal* ; 17 or 25 plays the 'character' of number of grandchildren; $\frac{4}{17}$, $\frac{4}{25}$ plays the 'character' of the amount of chaakkarapongal got by each child on the respective days.

This story indirectly throws light on the property:

"*A fraction with same numerator and increased denominator decreases in value*", in their thought process.

Children learn that it is also possible to compare fractions with same numerator and different denominators.

The story continues like this:

There also lived another Grandma named *Mookayee* in the same village in the neighborhood of *Raakayee*'s house. Her sons and daughters were also settled in towns, cities and even abroad.

Mookayee also didn't oblige to her sons or daughters invitation as she wanted to stay back at her beautiful village, just like *Raakayee.*

Her grandchildren also visit her during the summer holidays every year.

In one of these summers, there were 21 grandchildren present at the village.

Their Grandma *Mookayee* prepared *Chakkarapongal* in 4 identical pots fully.
She then served each of them equally.

Her grandchildren enjoyed feasting the sweet dish and thanked their grandma for the same.

They also told her that they would be expecting the sweet dish for the next day too.

Their grandma told them that she will be preparing the sweet dish every day as long as they were present.

All of them jumped with joy.

The very next day, two of her sons arrived at the place,

but no other grandchildren could accompany them. Now, the number of grandchildren remains the same.

Mookayee prepared *Chakkarapongal* (again!) in 5 identical pots fully, one pot more than the day before. She then served each of them equally.

Her grandchildren enjoyed feasting the sweet dish again and thanked their grandma for the same.

One of her granddaughters named *Uma* was present on both these days.

a) On which day, *Uma* would have got more *Chakkarapongal*?
[It should be the second day. Is n't it?]

b) Does this compare the two fractions $\frac{4}{21}$ and $\frac{5}{21}$?

Most of the people could answer (a) but many do not comprehend (b).

The reason is quite simple! Most of them are able to connect the numerals with the characters unknowingly. In this case, the numeral 4 or 5 plays the character of number of identical pots of *Chakkarapongal* ; 21 plays the character of number of grandchildren; $\frac{4}{21}$ and $\frac{5}{21}$ plays the character of the amount of chaakkarapongal got by each child in the respective days.

This part of the story indirectly throws light on the property: "*A fraction with same denominator and increased numerator increases in value*", in their thought process.

Thus it is also possible to compare fractions with same denominator and different numerators.

It is in fact an advantage for mathematics teachers to tell such well designed and articulated stories to the students. This will enable them to imbibe the mathematical idea or concepts through simple questions at the end of the story.

***Story 2:** Liza, Trisha **and** 'Pizza'*

There were two friends namely *Liza* and *Trisha*, studying in class 6 of the same school. They also have fun together. Both of them were fond of '*Pizza*'.

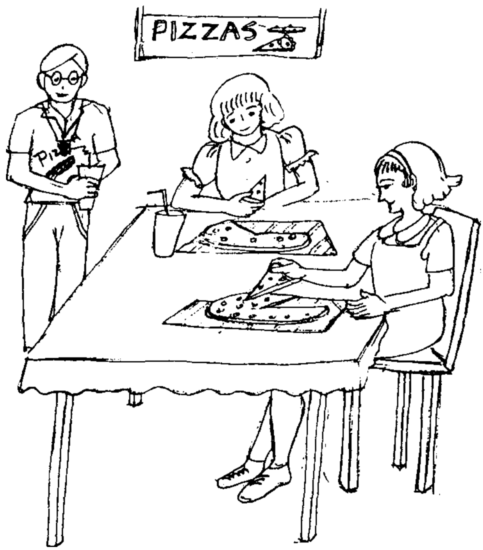
They used to visit the *Pizza* shop near the school, buy one *Pizza* and share it.

On a fine day, both went to the *Pizza* shop and *Liza* ordered as usual one *Pizza*.

Trisha interfered: "Come on! Let's try to have one *Pizza* each. It will be nice."

Liza asked her back: "Is it possible for us to eat one *Pizza* fully?"

Trisha replied: "I am not sure. But I believe it's worth trying."



They decided to buy two identical (same size) *Pizzas* and ordered the same.

They started eating their respective *Pizzas* but they couldn't eat fully.

a) If *Liza*'s leftover piece is larger than that of *Trisha*'s, who ate more *Pizza*?

b) Which is greater: $\frac{13}{17}$ or $\frac{21}{25}$?

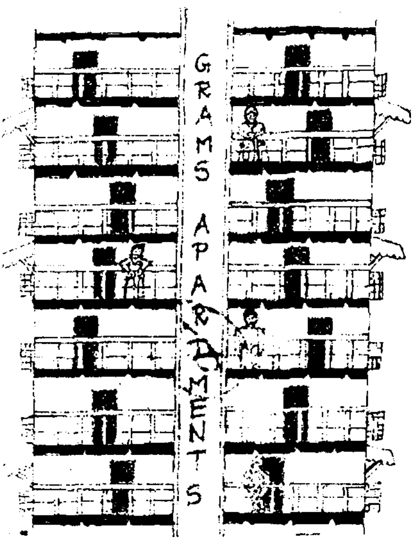
Again (a) is easy but (b) is somewhat perplexing.

	Ate	Left over
Person 1	$\frac{13}{17}$	$\frac{4}{17}$
Person 2	$\frac{21}{25}$	$\frac{4}{25}$

But if you presume that one of them ate $\frac{13}{17}$ of the *Pizza* and the other $\frac{21}{25}$ of the *Pizza*, then the left-over of Person 1 is more than that of Person 2. (Recollect *Raakayee* Story!)

Therefore, *Trisha* (person 2) ate more *Pizza* than *Liza* (person 1).

Story 3: Multi Storey Building



'Grams Apartments' was a multi-storey building. Among the residents who lived there, there were four friends *Ahmed, Bharghavi, Christie* and *Dharam Singh.*

Ahmed lives in one of the several floors of the building.

Bharghavi lives 2 floors above *Ahmed.*

Christie lives 1 floor above *Ahmed.*

Dharam Singh lives 2 floors below *Ahmed.*

a) Can you arrange these 4 persons in the ascending order of the altitudes in which these four persons live?

b) Is it necessary to know which person lives in which floor to answer the previous question?

c) Arrange the following fractions in ascending order: $\frac{17}{32}, \frac{19}{36}, \frac{13}{28}, \frac{21}{40}$

d) Is it easy to answer the above question if framed as follows?

Arrange the following fractions in ascending order:

$\frac{1}{32}$ more than half ; $\frac{1}{36}$ more than half ;

$\frac{1}{28}$ less than half; $\frac{1}{40}$ more than half

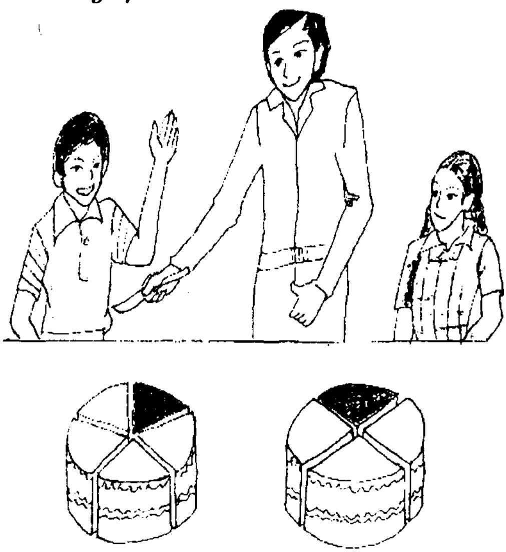
e) Why is the form of question (d) easier than the form of question (c)?

Most people could answer (a) and (d) easily not so easily (c).

The reason is quite simple! Reference for comparison is not zero (ground floor)!

$\frac{1}{2}$ plays the 'character' of reference number, like that of *Ahmed*'s floor number.

Story 4: Which cake to take? Which Piece to Leave?



Ramanujan, a '*mathe -matics professor*' celebrated his birthday. He cut two identical cakes, one into 4 equal parts and the other into 5 equal parts as shown. His two children, a son *Anshul* (11 years) and a daughter *Thejeswini* (10 years) were very fond of cakes and they were waiting to get from their father, their shares.

He said to his children: "I have a small puzzle for you. Suppose I allow you to choose one of these two cakes. You are allowed to leave a piece of the chosen cake and take all the remaining pieces. Which cake do you choose?"

Anshul and *Thejeswini* discussed for a while and came up with

the response: "We'll choose the cake that is divided into 5 equal parts and leave a piece so that we get 4 pieces."

Ramanujan asked: "Is it because you want to choose 4 pieces instead of 3?"

Anshul replied: "Not really. Some times 3 pieces may be large enough or equal to or smaller than 4 pieces. It all depends on the individual sizes of the pieces."

Ramanujan: "What is the reason for your choice?"

Thejeswini said: "When you divide a cake into more number of pieces, the size of each piece becomes smaller. Hence the size of the piece of 5-piece cake is smaller than that of the 4-piece cake. Therefore, it is better to leave a smaller piece so as to choose the remaining larger portion of the cake. That is the reason for choosing the 5-piece cake."

Ramanujan said: "Good! You are right. I just tested your mathematical I.Q. I'll give you 2 pieces each from the 4-piece cake and I, your mom and three of our friends will have 1 piece each from the 5-piece cake. You both can enjoy a larger and equal share." *Anshul* and *Thejeswini* jumped with joy.

a) Does this story tell you the fact $\frac{3}{4} < \frac{4}{5}$?

b) Can we extend this idea behind the story to establish the fact: $\frac{1}{2} < \frac{2}{3} < \frac{3}{4} < \frac{4}{5} < \frac{5}{6} < \ldots < \frac{99}{100} < \frac{100}{101} < \ldots$?

c) Can we further modulate the idea to establish the fact: $\frac{1}{3} < \frac{2}{4} < \frac{3}{5} < \frac{4}{6} < \frac{5}{7} < \ldots < \frac{98}{100} < \frac{99}{101} < \ldots$?

In fact, we may establish the powerful property:

"If both the numerator and the denominator of a proper fraction are increased by the same amount, the fraction also increases."

This property can also be revealed through the *Pizza* story. Verify!

Let us see how effectively one can use the four properties of fractions obtained (realized) from these four stories.

Property a) A fraction with same numerator and increased denominator decreases in value. (*Raakayee's story*)

Property b) A fraction with same denominator and increased numerator increases in value. (*Mookayee's story*)

Property c) If both the numerator and the denominator of a proper fraction are increased by the same amount, the fraction also increases. (*Cake / Pizza story*)

Property d) Fractions can also be compared from a different reference number rather than the conventional reference number zero. (*Ahmed's Multi-stor(e)y*)

Exercise: Compare the given fractions

1). $\frac{7}{23}; \frac{14}{45}$ 2). $\frac{29}{46}; \frac{41}{69}$ 3). $\frac{37}{108}; \frac{34}{99}$ 4). $\frac{27}{40}; \frac{38}{51}$ 5). $\frac{31}{81}; \frac{12}{37}$

6). $\frac{1237}{2340}; \frac{1236}{2341}$ 7). $\frac{19}{41}; \frac{60}{103}$ 8). $\frac{29}{53}; \frac{61}{19}; \frac{15}{31}$.

Solutions

1) $\frac{7}{23} = \frac{14}{46} < \frac{14}{45}$. Therefore, $\frac{7}{23} < \frac{14}{45}$.

Property a) is used. *Raakayee* is helpful here!

2) $\frac{29}{46} = \frac{87}{138} > \frac{82}{138} = \frac{41}{69}$. Therefore, $\frac{29}{46} > \frac{41}{69}$.

Property b) is used. *Mookayee* is helpful here!

3) $\frac{37}{108} = \frac{36}{108} + \frac{1}{108} = \frac{1}{3} + \frac{1}{108}$. $\frac{34}{99} = \frac{33}{99} + \frac{1}{99} = \frac{1}{3} + \frac{1}{99}$.
$\frac{37}{108} = \frac{1}{108}$ more than $\frac{1}{3}$. $\frac{34}{99} = \frac{1}{99}$ more than $\frac{1}{3}$.
Therefore, $\frac{37}{108} < \frac{34}{99}$.

Property d) is used. *Ahmed* is helpful here!

4) $\frac{27}{40} < \frac{27+11}{40+11} < \frac{38}{51}$. Therefore, $\frac{27}{40} < \frac{38}{51}$.

Property c) is used. Cake or *Pizza* is helpful here!

5) $\frac{31}{81} < \frac{24+7}{74+7} > \frac{24}{74} = \frac{12}{37}$. Therefore, $\frac{31}{81} > \frac{12}{37}$.

Property c) is used. Cake or *Pizza* is helpful here!

6) $\frac{1237}{2340} > \frac{1236}{2340} > \frac{1236}{2341}$. Therefore, $\frac{1237}{2340} > \frac{1236}{2341}$.

Properties b) & a) are used. Both *Mookayee* and *Raakayee* are helpful here!

7) $\frac{19}{41} = \frac{19+11}{41+11} = \frac{30}{52} = \frac{60}{104} < \frac{60}{103}$. Therefore, $\frac{19}{41} < \frac{60}{103}$.

Properties c) & a) are used. Cake and *Raakayee* are helpful here!

8) $\frac{15}{31} < \frac{15}{30} = \frac{1}{2} = \frac{24}{48} < \frac{24+5}{48+5} = \frac{29}{53} = \frac{58}{106} < \frac{58+3}{106+3} = \frac{61}{109}$.

Property c) is used. Cake is helpful here!

One may also use other ways (properties) to work out the same problems. Remember the fact "What is easy to compare with a particular property may be tedious with another. Easy or Tough is based on the data and the property applied. It cannot be generalized."

Note: In real life. Students aren't confused with different modes of transportation from school to home or vice versa.

T h e n ......

Why do we fix only method (LCM of denominators) to compare fractions? One may explore the treasure of several unknown properties of fractions. By telling math-story to the students and posing simple questions thereafter will enable the following:

(a) It allows the students to focus more, as they all like stories.

(b) It helps them to connect each figure with some character of the story.

(c) It helps them to understand concepts as well as find new techniques.

(d) It provides opportunity for the students to learn by themselves.

Teachers must feel the aroma (divergent thinking) from the natural and beautiful flower (mathematics). Allow the students to feel the same!

Special Thanks to Mrs Saraswathy Girish, specialist artist who has helped me in providing nice and relevant pictures for the article.

USEFUL TIPS FOR INTRODUCING SOME MATHEMATICAL CONCEPTS

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Introduction:

While studying mathematics, most of us get lost in the symbols, notation and the steps of the proof of a theorem. Many text books do not address the difficulties faced by the students. Text books should present the topics in their historical perspective and include a number of illustrative sketches which can go a long way in improving the understanding of the subject by the students. The teacher can make the class lively by giving a number of illustrations from everyday life and by giving historical notes. In this article, the author shares some of his interactions with colleagues and students and his own learning experience.

Complex Numbers:

Introducing complex numbers to a class is really a complex task! Many students are at a loss to understand how 'imaginary' numbers can represent real physical quantities! The teacher should dispel all the misconceptions from the minds of the students in the beginning itself by saying that a complex number is a pair of real numbers! A complex number has two components which we may call as the first and second components (instead of the usual terminology of real and imaginary parts, which causes so much confusion in the young minds!).Both the components are real and there is nothing

fictitious or imaginary or mystic about them! Historically, the complex number system originated from attempts to solve the equation $x^2 + 1 = 0$. By writing $i^2 = -1$, we have not found a solution within the real number system, but have succeeded in creating another number system of which the real number system can be considered as a subset. The teacher may drive home the point that the actual meaning of the equation $i^2 = -1$ is that the product of the two complex numbers $(0,1) \times (0,1) =$ the complex number $(-1,0)$. In the same way, the expression $a+ib$ is the simplified form of the complex expression $(a,0) + (0,1) \times (b,0)$.

The teacher may add that the complex number system is very useful whenever we want to study a pair of physical quantities together e.g., voltage and frequency of an alternating current or the velocity components of an ideal fluid flow. Apart from being a two dimensional vector space of real numbers, the complex number system has an additional advantage of possessing an algebraic (field) and metric (distance) structure (similar to the real number system). The theory of analytic functions of a complex variable provides the base for interesting applications in physics and engineering.

Vector Algebra and Analysis:

In the higher secondary school, a vector is introduced as something having magnitude and direction. In the later stages in college, many students have difficulty in understanding the concept of vector spaces. They still think in terms of magnitude and direction. Very often, students mistake the word 'space' for the physical space around us! The teacher may clarify that this is purely a mathematical concept and he may even use the word 'manifold' or 'ensemble'. The teacher must emphasize that though the basic axioms governing vector algebra are taken from physics, (like addition of velocities, accelerations, electro-

magnetic forces etc) the scope of vector algebra and analysis goes beyond these physical systems and can encompass much wider systems. Indeed, this is true of all mathematical systems! The teacher should give a number of examples of scalar point functions (e.g. temperature and pressure distribution in the atmosphere), vector point functions (e.g.position and velocity of a moving aircraft, missile or rocket). Vector algebra and analysis are useful whenever we want to study a set of objects (physical or otherwise) together. The teacher may emphasize the following important points.

The vector concept gives us a precise and concise way of describing any multi component system.

The vector concept helps us to formulate the laws of a system, independent of the reference co-ordinate system.

The vector notation is similar to that used in the ordinary arithmetic or algebra (e.g. $A + B$)

The governing axioms of vector addition are similar to those in ordinary arithmetic or algebra, but the axioms governing vector product (cross product) are not similar. Here the teacher may draw a distinction between dot product (scalar product) and cross product. (There is no unit vector for vector products, there is no cross product inverse and the associative and commutative laws do not hold for vector products).

With the use of vector concept and notation, generalization from one dimension to several dimensions is easy and straight forward. The teacher can illustrate how the ideas of differentiation, integration, Taylor series expansion and differential equation for a function of a single variable can be generalized for a vector function.

These ideas can help a student to comprehend multi-dimensional vector spaces, like the 4 dimensional space-time

continuum used in the theory of relativity. Again, the teacher should emphasize that there are multi dimensional manifolds which admit an algebraic (field or division ring) structure, with element multiplication. This multiplication is different from the dot product or the cross product. R^2 (the complex field) and R^4 (the division ring of quaternions) are examples. These manifolds also have interesting real life applications.

Stokes Theorem, Gauss Divergence Theorem

Many students have difficulty in understanding these theorems, which have a number of applications. Even before they get into the maze of the various steps of the proof, it is important that the students get a feel of these theorems. The teacher may start from the fundamental theorem of calculus which says that if $f(x)$ is continuous, $\int_a^b f(x)dx = F(b) - F(a)$ where $dF/dx = f(x)$. The value of the integral depends only on the value of $F(x)$ at the end points of the interval : a and b. Stokes theorem says that $\int_s$ curl $fdS = \int_c f.dr$, where C is the bounding curve of the surface S. Here again, this result depends only on the rim C and is independent of the surface which passes through C. This is due to the fact that the integrand is in some sense the derivative of f and the integral becomes exact and needs to be evaluated only along the rim. Thus a surface integral is reduced to a line integral. A similar argument holds for the Gauss divergence theorem which says that $\int_v div\, f dv = \int_s f.dS$. Here again, the value of the integral depends only on the bounding surface S. In this case, a volume integral is reduced to a surface integral. Stokes theorem reduces to Green's theorem for a plane, which says $\int_A$ curl $f.kdA = \int_c f.dr$. An area integral reduces to a line integral, under conditions of continuity of the integrand.

The teacher may illustrate the close link between two dimensional vector analysis and the theory of complex analytic

functions. In the case of complex analytic functions, Cauchy's fundamental theorem on analytic functions says that $\int_C f(z)dz = 0$, where C is any simple closed curve. This result can be proved by invoking Green's theorem, Cauchy-Riemann equations and by assuming continuity of $f'(z)$. The teacher may emphasize that the Cauchy-Riemann equations are based on the path independence principle of the complex derivatives. It is interesting that Cauchy-Riemann equations make the left hand side of Green's theorem zero and thus making the integral of f also path independent! The Cauchy-Riemann equations also indicate that analytical functions arise as the gradient of certain potential functions. Incompressible and irrational flows can thus be represented by complex analytic functions. Since the integrals of analytic functions are path independent, they can represent conservative force fields. These arguments can give the students a lot of feel for these results.

The teacher may also give physical interpretation of div f, curl f etc. For example, div f represents the net mass flow or heat flow from a control volume in the heat conduction or fluid flow equation.

Maxima and Minima

This is an example to show how generalization of ideas takes place in mathematics. In the case of functions of a single variable, $f(x)$ attains an extremum when $f'(x) = 0$. This gives an algebraic equation for x. $f(x)$ is minimum or maximum depending on whether $f''(x) > 0$ or $f''(x) < 0$. In the case f is a scalar point function (scalar function of a vector variable), the condition for extremum is grad $f = 0$ and f is maximum or minimum depending on whether the Hessian matrix of f (the matrix of second order derivatives) is negative or positive definite. In the case of functional, (functions of

functions) the optimum f is found out by solving a differential equation in f. (Euler-Lagrange equation).

Inter-relation between Geometry and Mechanics

Many problems in Mechanics and Physics give rise to interesting ideas in geometry and vice versa. When the equation of a curve is expressed in parametric form and if that parameter is taken as time, we get an equivalent problem in Mechanics. These inter relations not only give us better insight into certain theorems, but they suggest the method of proof also. The teacher may cite the following examples :

The centripetal force experienced by a particle moving along a curve is related to its radius of curvature.

The Brachistorone (line of quickest descent) led to the discovery of cycloid.

The path of a particle moving on a surface only under the force of normal reaction is a geodesic.

The geometric shape of a suspension bridge is a catenary.

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A NOTE ON THE CIRCUM CIRCLES ASSOCIATED WITH A RECTANGLE

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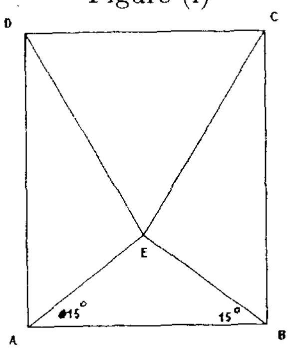
The following two results were motivated by a problem that appeared in an AMTI competition in the late 70's. The problem was to find tan 75° geometrically. The following are the two results motivated by a solution to the above problem.

Result I

Let $ABCD$ be a square. Let E be a point inside $ABCD$ such that $\angle EAB = \angle EBA = 15^\circ$ as shown in figure (i) below. Then CED is an equilateral Δ.

Figure (i)



Result I is an exercise in the book 'Geometry revisited' by Coxeter and Greitzer published by the Mathematical association of America. The authors hint at an indirect proof at the back of the book. What is new here is that we give a simple direct trigonometric proof of Result I using the value of tan 75° and tan 15°.

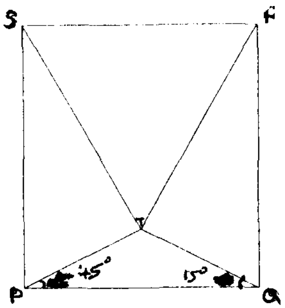
Result II

Let $PQRS$ be a rectangle such that $PS = 2PQ$.

Let T be a point inside the rectangle such that $\angle TPQ =$

$\angle TQP = 15^\circ$ as in figure (ii). then Δ's STP and RTQ are right triangles and the circum circle, of Δ's PTQ, TQR, STR and PTS are equal (equal in radius).

Figure (ii)



We first give a solution to the problem that appeared in the AMTI competition. Although it may have appeared in an earlier issue of the mathematics teacher we give it here for completeness.

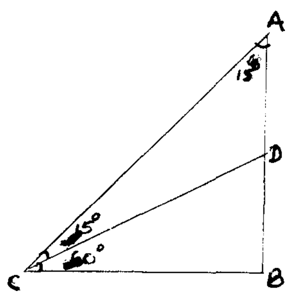
Let ABC be a right angled Δ, right angled at B.

Let $\angle C = 75^\circ$. Cut off $\angle BCD = 60^\circ$ (refer to figure (iii) below). Let $BC = a$ then $CD = 2a$.

And $BD = \sqrt{3}a$. Also since $\angle ACD = \angle DAC = 15^\circ$. $AD = CD = 2a$.

$\therefore\ AB = 2a + \sqrt{3}a = a(2 + \sqrt{3})$.

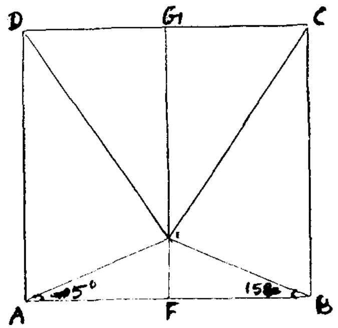
Figure (iii)



$$\tan 75^\circ = \frac{AB}{BC}$$
$$= \frac{a(2+\sqrt{3})}{a} = 2 + \sqrt{3}$$
$$\tan 15^\circ = \frac{AB}{BC}$$
$$= \frac{a}{a(2+\sqrt{3})}$$
$$= \frac{1}{2+\sqrt{3}} = 2 - \sqrt{3}.$$

Solution to Result I

Figure (iv)



We refer to figure (iv). Draw $EF \perp$ to AB and produce FE to meet DC in G. Let $AB = AD = 2a$. Then $AF = FB = DG = GC = a$. Since $EF = AF \tan 15°$ we have $EF = a(2 - \sqrt{3})$

And $EG = FG - EF = 2a - (2a - \sqrt{3}a) = \sqrt{3}a$

$\therefore$ In right triangle $DGE, DG = a \quad GE = \sqrt{3}a$.

$\therefore \angle GDE = 60^{\circ}$.

Similarly $\angle GCE = 60°$. $\therefore \Delta DEC$ is equilateral.

The above solution although uses trigonometry, can be considered geometrical since the value of $\tan 15°$ used in the proof was obtained geometrically. not sacrificing the spirit of 'Geometry revisited'.

Solution to Result II

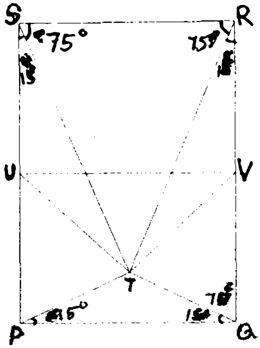
We deduce result II from result I.

Let U and V be the mid points of PS and QR join UV. (Refer to fig.(v))

Now since $PQVU$ is a square, we have from result I that UTV is an equilateral Δ.

$\therefore UT = UV = TV = PU$.

Figure (v)



In ΔPTS, U is the midpoint of PS.

And $UT = PU = US$.

$\therefore$ ΔPTS is right angled and $\angle STP = 90°$.

Similarly, ΔRTQ is right angled and $\angle RTQ = 90°$.

Since $PS = QR$, the circum circles of ΔPTS and ΔRTQ are equal.

Also the common chord TQ of circum circles of Δ's PTQ and TQR subtends equal angles of $15°$ at P and R on the circumferences of the respective circles.

$\therefore$ These circum circles are equal.

Also since ΔSTP and ΔTQR are congruent we have $ST = TR$ and $\angle TSR = \angle SRT = 75°$.

A similar argument as above shows that the circum circles of Δ's STR and TRQ are equal and we have the desired result.icular, a rectangle is always a

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46TH ANNUAL CONFERENCE OF AMTI

Gist of Panel Discussion

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Introduction

Panel discussion was arranged as a part of the academic programme during the 46th Annual conference of the Association of Mathematics Teachers of India (AMTI) held at Baramati, Dist. Pune (Maharashtra State) from December 27 to 29, 2011. The discussion focused on the main theme of the conference "Constructivism in Mathematics Education." Constructivism is a new concept that is being advocated by the National Curriculum Framework 2005 for school education. It is based on the premise that every student makes meaning of different experiences given to them and construct his/her own knowledge. So far, our school teaching was based on the assumption that students who come to school are like clean slate, they receive and store the information given to them in classroom interaction. We made them to listen, repeat and reproduce when required. Those who fitted into this pattern were rewarded by good grades in school examinations.

New Thinking in educational psychology, however refutes, this idea and tells us that the learner is not a reviver but a constructor of knowledge. If we decide to follow this mode of thinking our entire method of providing inputs to school children needs to be changed. Moreover, the role of the a

teacher also changes drastically. Realising the need of educating the conference participants with this new idea of constructivism in school education in general and in mathematics education in particular, the panel discussion was planned during the conference.

Panel Discussion

The panel consisted of the following members.

1. Dr. Bhaktavatsulu, Chennai, Tamilnadu
2. Shri Bhat, Pune, Maharashtra
3. Shri Govind Reddy, Nandyal, Andhra Pradesh
4. Dr. S.B Deosarkar, Baramati, Maharashtra

Dr. Sudhakar C Agarkar, Professor, Homi Bhabha Centre for Science Education, TIFR, Mumbai monitored and coordinated the discussion. For convenience each participant was asked to focus on one of the perspectives. Accordingly, Dr. Bhaktavatsulu concentrated on teacher's perspectives, Shri Bhat focused on Pupils' perspectives, Dr. Deosarkar gave his comment through interaction perspective and Shri Govind Reddy put forth the global perspectives. The summary of all these perspectives prepared by Prof Agarkar is given below for the convenience of readers of Mathematics Teacher.

Teachers' Perspectives

1. In the new paradigm of constructivism many of the practising teachers are unable to understand their role clearly. They were comfortable in playing the role of knowledge dissemination effectively. However, they found themselves lost in the woods when they were asked to use constructivism approach of teaching in the classrooms.
2. In the new paradigm the teacher is seen as a facilitator in the construction of knowledge by the students. They need

to be trained before they can play this role effectively. There is a widespread feeling that practising teachers have not been adequately prepared to implement this new pedagogy effectively.

3. If the student is to construct his/her own knowledge he/she needs to be given adequate experiences and study material. There is a scarcity of quality material that would encourage construction of knowledge. The task of preparing suitable supporting material for education is, therefore, need of the day.

4. Teacher must realise that a Child-child interaction plays a major role in the construction of knowledge. The present classroom interaction hardly provides an opportunity for such an interaction. Arrangement has to be made to enhance this interaction by forming small groups and engaging them in some useful assignments or the other.

5. Teacher must understand that students would be motivated to learn if the examples from their everyday lives are taken for discussion. Identification of such situations where school mathematics is used in everyday life needs to take place on priority basis.

Pupils' Perspectives

1. As soon as the child enters the school he/she is asked to keep quiet and to listen. He/she is so conditioned to listening and note taking that many of the students are unwilling to undertake an activity involving exploration. First this conditioning of listening and remembering has to be removed. This task is not that simple as it may appear at first glance.

2. In order to engage the student actively in the process of learning it is necessary that they are exposed to cognitive

conflict. Once they come across such situation they will try to resolve it and in turn learn about the empathetical concept involved.

3. Co-creation is the key aspect of constructivism. It is the cooperation among the students and not the competition that would lead to better learning. In the present scenario the competition dominates over cooperation. It is a big challenge to reverse this order.
4. In case of mathematics, problem solving plays an important role in concept formation and concept fixation. Ample opportunities need to be created for solving challenging problems instead of doing the same repeatedly which causes boredom among the students.
5. Creation of an environment of learning through constructivism would be liked by the school students as it is their natural way of learning. It is, therefore, necessary to undertake it without any further delay and students be made responsible for their learning.

Interaction Perspectives

1. In the context of interaction perspective three types of interactions were considered: teacher-pupil interaction, pupil-pupil interaction and pupil-family interaction. Children learn through all these interactions and they get influenced by adopting constructivism approach of teaching.
2. Teacher taught relation between teachers and students changes into facilitator learner relation. Teacher has to facilitate construction of knowledge instead of spoon feeding the existing information.
3. Peer interaction assumes an important role in the new model of teaching. Cooperative learning becomes the

essential aspect of child-child interaction to enhance learning.

4. Technology has witnesses unprecedented developments in the recent past. It has changed the way pupils interact with each other and the way they gain knowledge. This technology should be put to use to enhance exchange of information and experiences among the students.
5. So far, society has remained aloof from school education. Such aloofness on the part of the parents and society will not serve the purpose. A school child spends a lot of time with parents and other members of the society. The experiences gained in the social life also shapes the personality of school children. The society, therefore, has the responsibility to ensure that proper inputs are given to child so that he/she acquires right knowledge and values.
6. Family has a major role to play in the knowledge growth of the child. The inputs that the child gets from home, the nature of interaction he/she has with family members are similarly important. These experiences are to be coupled with the school experiences to ensure that there is no conflict between these two and they go hand in hand.

Global Perspectives

1. Developments in Information and Communication Technology (ICT) has made the entire world a global village. In this technological world information from one place moves to another place quite fast. The child, a citizen of the next generation, needs to be trained to filter relevant information and use it for the benefit of his/her society.
2. The teaching profession has now become the wage earning profession all over the world. Teachers are hardly

concerned with the overall development of the child. They focus on passing the information that is assigned to them. Such a narrow view will not work in the fast changing global society as the person studying in one country will have to serve another country.

3. The curriculum developers look at different subjects differently. As a result, the fragmented information is passed on to the students. They must be given a gestaltic view of the world and its problems.
4. There is a general saying that a person who can do nothing else enters into teaching profession. Society looks at it as a profession needing low level of intelligence and managerial skills. As a result, there is gross dissatisfaction among the teaching community. This perspective possessed by the society needs to be changed immediately. Teaching profession must be considered as a respectable profession by the society.
5. Society is changing fast and its demands from school education are also changing There is a tremendous growth of knowledge. In order to keep pace with this growth the teacher has to be a life long learner. Moreover, he/she has to inculcate the skills of learning among the students so that they also become life long learners. Learning to learn is the skill that will remain with them for their entire life.

Coordinators comments

1. Constructivism is now a buzz word in education. It is a word coined in the west and borrowed by us. Nevertheless, we are not totally new to this idea of constructivism. If we look to our ancient method of teaching in gurukul system, we find that the similar method was being used there. There is a Sanskrit saying

that says a person gets knowledge through four different sources: About one fourth of his knowledge he gets from his guru (teacher), about one fourth he gets from peers, about one fourth he gains through daily experiences and about one fourth he gains through his own efforts. One notices that a child in ancient system was given all these four opportunities to gain knowledge. Students in ashram schools interacted with their teacher and gained some knowledge. They also worked in teams, interacted with peers and gained knowledge. They gained part of he knowledge through undertaking daily activities of the Ashram like grazing a cow, milking it, selling milk and curd in nearby villages. They also attempted to enhance their knowledge by contemplating over events or by conducting a dialogue with experts.

2. The present method of school education is the direct outcome of industrial revolution. Schools were started to fulfil the demands of industries. As the industrial revolution has been replaced by Information revolution the demand has changed substantially. In these days one expects to create leaders with Contextualised Multiple Intelligence (CMI) to deal with the problems of the society. Our efforts to create a generation that would follow instructions is not going to take us far. Instead, the CMI leaders would make our country progress in all respects. The time has come to look at school education with new perspectives through constructivism.

REPORT ON THE 46TH ANNUAL CONFERENCE OF THE ASSOCIATION HELD AT VIDYA PRATISHTHAN, BARAMATI

The conference was inaugurated on 26th December 2011 at 10am. The inaugural function was attended by Dr. S.R.Bhonsle, formerly of Michigan Tech. University as chief guest and officials of Vidya Pratishthan. The chief guest released the souvenir and inaugurated the conference. The general secretary of the AMTI, Sri M.Mahadevan, presented the annual report. Executive Chairman, Prof. J. Pandurangan proposed the vote of thanks. Earlier the children of New Bal Vikas Mandir sang the prayer song specially composed by Sri Bharat Karmarkar with slokas on the glory of Mathematics and Mathematicians and Prof. Deo Sarkar, principal, COE, V.P.Baramati welcomed the gathering.

After tea, delegates reassembled, when the executive chairman presented the address of the President, Prof. Rajendra Bhatia, who could not join the conference due to other engagements.

Then Dr. Smt. Pawar. Y.S., Prof. from Kolhapur University gave the theme talk on Constructivisim in Mathematics after which Lunch was announced.

In the post lunch session there were paper presentations first by teachers followed by students in the common hall to to facilitate interaction. After tea the teaching aids exhibition was inaugurated by the Executive Chairman followed by entertainment by the children of Vidya Pratishthan schools depicting the culture of Maharashtra, which was enjoyed by all the delegates. Then dinner was announced ending the programme of the first day.

The second day started with written quiz in which 60% of delegates took part. This was followed by Prof. A.Narasinga Rao Memorial Lecture delivered by Prof. K.N.Ranganathan from Chennai, retired professor from Vivekananda College and regional coordinator, TN & Puduchery for INMO. Prof. J. Pandurangan chaired the session. After tea Prof. P.L.Bhatnagar memorial lecture was delivered by Dr. Arun Adsool, former V.C., Pune University and Principal, Vidya Pratishthan Arts, Com. & Sc. College. Prof. Deo Sarkar, chaired the session.

Then followed paper presentations by teachers in three halls, to accommodate more aspirants to present their papers. After that lunch was announced.

In the post lunch session Prof. R.C.Gupta Endowment Lecture was delivered by Prof. S.S.Sane, National Co-ordinator for INMO. He spoke on Fibonacci numbers. Prof. Pandurangan was in the chair.

Then started the panel discussion on the theme of the conference in which Sri Bhat, Dr. P. Bakatavatsulu, Sri Govinda Reddy and Prof. Deo Sarkar participated as panelists with Prof. S.C.Agarkar moderating. Each represented respectively students' perspective, teachers' perspective, social perspectives and students' and teachers' perspectives.

After tea delegates were taken for local sightseeing while the general body of the AMTI met and deliberated. With dinner the second day came to an end.

The third day started with students presenting their papers in three halls where this year more children got chance to present their papers. They were from class 7 to college level students. After tea selected teams, four in number, took part in the oral quiz competition with Sri R. Athmaraman as

quiz master ably assisted by Prof. Shanti and Sri Venkata Subramaniam. Then lunch was announced.

In the post lunch session there were short communications by delegates including brief presentation of their papers by two. Then started the valedictory function in which Prof. S.C. Agarkar welcomed the gathering. From north, east, west and south representatives gave their impressions on the conference including the confidence generated in the students for public performance as a positive contribution of the AMTI conferences besides their know how improving by interaction and sessions. After the valedictory address by Dr. Murunkar, Principal of T.C. College, Baramati, the general secretary proposed vote of thanks thanking all the personnel involved from Vidya Prathishthan for the wonderful hospitality, technical and social help extended to one and all of the delegates who attended the conference. With expected venue for 47th conference announced as Visakhapatnam(?) and national anthem the curtain came down the 46th conference of the association.

Packed dinner and distribution of certificates was also completed well ahead of schedule enabling out-station delegates to leave without tension.

A visitor's impression on the conference received by e-mail on 17th February 2012

Mahadevan,

I am just now getting around to thanking you for the wonderful time I had attending the math conference on December 27th. The people I met, the discussions that took place, the presentations that were made, the information which was shared, all gave me a very good understanding of math instruction in India as well as the firm belief that we face

many of the same problems and challenges making instruction relevant, rigorous and applicable in a quickly changing world. I also found the discussion of India's place in the history of math was the very subject on which I was centering my trip to India, particularly to Gwalior. I was able to see the tablet at the Gwalior Fort which is the earliest use of zero the way it is used today. I have attached a picture of the tablet, as well as a picture of one of the rubbings I took from it as a sample and keepsake of my visit.

Thank you again, and please let me know if there is anything I can do for you from here in Virginia, USA.

All the best.

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ASSOCIATION ACTIVITIES

1. This year in Grand Achievement test for class XII students of Tamil Nadu, 4506 from 48 schools participated. Ten were given silver medals. 150 were given cash award of Rs.100/- each with certificates and certificate alone for getting 90% and above for 316 were distributed besides claims of evaluations, well on time.

2. For the year 2010-11 accounts were audited and approved by the general body in its meeting on 28.12.2012.

3. The building up to ground floor in our plot will be ready by May in which we propose to organize our teachers' workshop this year. Sponsors and interested teachers may contact us latest by 01-05-2012. This will help us plan the workshop better commensurate with the demand.

4. Call notices for NMTC 44 will be ready be 1st week of June formally. Prospective participants individuals, institutions may see our website, download the proforma, fill in and send the data to reach us latest by 31.07.2012. ***Earlier we used to accept letters also but from this year entries without the proforma issued by us may not be accepted as also delayed receipts.***

5. There will be slight modification in the Sunday workshop. Please see our website by the end of April to get the details.

6. (i) The 14th popular lecture was delivered by Sri. C.V. Narasimhan, IPS and former CBI Director on Mathematical Perspectives in Crime Investigation.

(ii) 15th popular lecture was given by Sri S Vesley Calvert, Dept. of Maths, Southern Illinois University on paradox, truth and computing.

7. (i) Two Teachers' Workshops at Vidya Pratishthan, Baramati for 6 days, each of 3 day's duration, were conducted separately for primary and secondary teachers in the first work of November 2011.

(ii) A Five day ESTEEM workshop for JNV teachers of southern region was conducted with 31 participants from 5th to 9th September 2011, at IMSc., Taramani, Chennai.

(iii) A two day workshop for JNV students of southern region was held at JNV, Maddirala with Sri. Sadagopan Rajesh, as the only resource person.

(iv) Six teachers of P.S.B.B. School and the General Secretary attended a two day workshop organized by ISO, Bangalore on 7th and 8th January 2012.

(v) In connection with the Rashtriya Madhyamik Shiksha Abhiyan workshops for mathematics teachers, the AMTI was contacted from two district authorities one at Tiruvarur (14.02.2012) and the other at Chennai (20.02.2012 to 24.02.2012). Our general secretary and at least three other resource persons guided the teachers for improvement of mathematics performance in state schools - Govt., aided and corporation schools.

8. Representatives of AMTI were invited to National Initiative for Mathematics Education (NIME) held at COCHIN (November 2011) and MUMBAI (20 to 22nd January 2012) where the services of the AMTI were presented through power point projection.

9. On 25-02-2012 the general secretary was invited for a brain storming session in NCERT for celebration of National Mathematics Year 2012.

10. For the DST organized Vigyan Prasar Conference on popularizing Mathematics also the general secretary was invited and the role of AMTI in that connection was disseminated.

11. The AMTI has announced workshops for Teachers and Students in the upcoming new building at Alappakkam. Registrations are on.

M.Mahadevan
General Secretary

NATIONAL MATHEMATICS YEAR

We shall all join the plan to celebrate

National Mathematics year 2012

by organizing

meetings, exhibitions, talks, honoring local mathematicians

from now till **22nd December 2012.**

In honoring

SRINIVASA RAMANUJAN

we honour ourselves and our country.

Please send brief reports on the activity to the AMTI by mail for suitable dissemination.

General Secretary

The Association of Mathematics Teachers of India

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